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Editorial

Dear CIM Colleagues,

On behalf of CIM, I would like to invite you to participate in the forthcoming CIM-MPE events being organized as part of the year-long, global celebration — Mathematics of Planet Earth 2013 (MPE-2013). These events are enthusiastically supported by many Portuguese institutions, including: SPM; SPE; APDIO; CEMAPRE; CEAUL; CMA-UNL; CMAF-UL; CMUP; INESC-TEC; ISR; IT; UECE; Fernando Santos Sucessores Lda; FCUL; ISEG; Calouste Gulbenkian Foundation and Ciência Viva.

One CIM-MPE event is the international conference, Planet Earth, Dynamics, Games and Science, which will be held September 2–4, 2013 at the Calouste Gulbenkian Foundation. In addition to the conference an advanced school will be held August 26–31 and September 5–7 at the Instituto Superior de Economia e Gestão ISEG in Lisbon, Portugal.

Distinguished Keynote speakers and lecturers will include: Eric Maskin, CIM-SPM Pedro Nunes Lecture, Institute for Advanced Studies, USA (scheduled); Michel Benaim, Université de Neuchâtel, Switzerland; Jim Cushing, University of Arizona, USA; João Lopes Dias, Universidade Técnica de Lisboa, Portugal; Pedro Duarte, Universidade de Lisboa, Portugal; Diogo Gomes, Universidade Técnica de Lisboa, Portugal; Yunping Jiang, City University of New York, USA; Jorge Pacheco, Universidade do Minho, Portugal; David Rand, University of Warwick, UK; Martin Shubik, Yale University, USA (by video); Satoru Takahashi, Princeton University, USA; Marcelo Viana, Instituto de Matemática Pura e Aplicada IMPA, Brazil

If you are interested in being involved in this conference, CIM is accepting proposals for thematic sessions. Proposals should include a session title and names and affiliations of 3–5 proposed speakers. If you are not interested in organizing a full session, but would like to give a presentation, send an email with your presentation title and abstract for consideration. Session proposals and presentation applications should be sent to aapinto@fc.up.pt and info.mpe2013@sqig.math.ist.utl.pt, by June 30.

Please remember to visit the event website to register: http://mpe2013.org/workshop/dgs-2013-international-conference-and-advanced-school-planet-earth-dynamics-games-and-science-portugal-26-august-to-7-september-2013/

The above conference and advanced schools are part of an ongoing series of events for MPE 2013, organized by CIM for Portugal. In March, CIM hosted the first CIM-MPE events — the international conference and advanced school, *Planet Earth, Mathematics of Energy and Climate Change*. Attendees at the opening ceremony were honored with the participation of the Honorable Mr. Secretary of State Professor João Queiró, FCT Board Member Professor Paulo Pereira, Portuguese Mathematical Society President Professor Miguel Abreu, Portuguese Statistical Society Board Member Professor Manuela Neves, MPT 2013 Executive Committee Mem-

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ber Professor Adérito Araújo, CMAF President Professor Luís Sanches, CMA Member Professor Fábio Chalub, Dom Luiz Institute Director Professor Pedro Miranda and the former CIM Presidents Professor José Perdigão Dias da Silva and Professor José Francisco Rodrigues.

During the Opening Ceremony, the President of CIM, together with Secretary Queiró, awarded the CIM Medals to the following distinguished recipients: José Perdigão Dias da Silva, Universidade de Lisboa; L. Trabucho de Campos, Universidade Nova de Lisboa; Joaquim João Júdice, Universidade de Coimbra; José Francisco Rodrigues, Universidade de Lisboa

The CIM Medals are awarded to mathematicians in recognition of meritorious contributions made during their scientific careers and to acknowledge their significant influence on the development of Mathematics in Portugal through affiliation with CIM.

In addition to those who participated in the opening ceremonies, CIM would like to thank the following keynote speakers and lecturers for their wonderful presentations: Richard James, CIM-SPM Pedro Nunes Lecture University of Minnesota, USA; Inês Azevedo, Carnegie Mellon University, USA; Christopher K. R. T. Jones, University of North Carolina, USA; Pedro Miranda, Universidade de Lisboa, Portugal ; Keith Promislow, Michigan State University, USA; Richard L. Smith, University of North Carolina, USA; José Xavier, Universidade de Coimbra, Portugal; David Zilberman, University of California, Berkeley, USA

CIM also appreciates the 60 invited speakers for their enlightening presentations and sincerely thanks the session organizers for their effort, commitment and dedication that was so vital for the success of the events: Ivette Gomes, Universidade de Lisboa; Stéphane Louis Clain, Universidade do Minho; Carlos Ramos, Universidade de Évora; Miguel Centeno Brito, Universidade de Lisboa; Raquel Menezes, Universidade do Minho; José Luís dos Santos Cardoso, Universidade de Trás-os-Montes e Alto Douro; Mário Gonzalez Pereira, Universidade de Trás-os-Montes e Alto Douro; Patrícia Gonçalves, Universidade do Minho; João Gama, Universidade do Porto; Delfim F. M. Torres, Universidade de Aveiro; Paulo A. V. Borges, Universidade dos Açores; João Paulo Almeida, Instituto Politécnico de Bragança; Alberto Adrego Pinto, Universidade do Porto ; Nico Stollenwerk, Universidade de Lisboa; Tânia Pinto Varela, Universidade Técnica de Lisboa; Maria da Conceição Carvalho, Universidade de Lisboa; António Pacheco Pires, Universidade Técnica de Lisboa; Margarida Brito, Universidade do Porto; Sílvio M.A. Gama, Universidade do Porto; João Emílio Almeida, Universidade do Porto.

In addition, CIM especially thanks Irene Fonseca for her scientific guidance, Antónia Turkman for her assistance with the Calouste Gulbenkian Foundation, Telmo Parreira for organizing and compiling the proceedings, and Paulo Mateus, Pedro Baltazar and Telmo Parreira for developing and maintaining the conference website. CIM thanks the CGF staff and members of the local organizing committee (Alberto Pinto, FCUP; Paulo Mateus, IST; Pedro Baltazar, IST; Telmo Parreira, FCUP; Abdelrahim Mousa, FCUP; João Almeida, IPB; Renato Soeiro, FCUP; João Coelho, FCUP; Bruno Neto, FCUP; Filipe Martins, FCUP; Joana Becker, FCUP, Renato Fernandes, FCUP, Ricardo Cruz FCUP), as well as the Calouste Gulbenkian Foundation, for incredible hospitality, throughout the event and for providing the speakers and participants with the opportunity to experience the friendly ambiance in the beautiful city of Lisbon.

Finally, CIM would like to remind you that review and research articles related to the two conferences will be published in the first two volumes of the *CIM Series in Mathematical Sciences*, which will be published by Springer-Verlag. The editors, Jean Pierre Bourguignon, Rolf Jeltsch, Alberto Pinto and Marcelo Viana, invite conference participants to submit review and research articles for consideration to aapinto@fc.up.pt and info.mpe2013@sqig.math.ist.utl.pt by December 31, 2013.

Alberto Adrego Pinto
President of CIM

Editors
Adérito Araújo (alma@mat.uc.pt)
António Fernandes (amfern@math.ist.utl.pt)
Sílvio Gama (smgama@fc.up.pt)

Address
IIIUL-Universidade de Lisboa
Av. Prof. Gama Pinto, 2
1649-003 Lisboa

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Coming Events

92nd European Study Group with Industry 2012
May, 6–10, 2013

ISEC—Institute of Engineering, Polytechnic of Coimbra, Coimbra—Portugal
http://dfm.isec.pt/esgi92/

The 92nd European Study Group with Industry will be held from May 6 to May 10 2013 at ISEC (http://www.isec.pt/), the Coimbra Institute of Engineering, Polytechnic of Coimbra, Portugal, organized by the Department of Mathematics and Physics, ISEC-DMF (http://www.dfm.isec.pt/) and LCM – Laboratory for Computational Mathematics (http://www.uc.pt/uid/lcm) of Centre for Mathematics of the University of Coimbra (https://cmuc.mat.uc.pt/rdonweb/).

This meeting is part of the series of European Study Groups and will count with the participation of several European experts with a large experience in this type of events. The purpose of these meetings is to strengthen the links between Mathematics and Industry by using Mathematics to tackle industrial problems, which are proposed by industrial partners.

More information on Portuguese Study Groups is available at http://www.ciul.ul.pt/~freitas/esgip.html, while general information on study groups and related aspects is available at the International Study Groups website (http://www.maths-in-industry.org), the Smith Institute (http://www.smith-inst.ac.uk) and the European Consortium for Mathematics in Industry (http://www.ecmi-indmath.org/info/events.php).

Ist Portuguese Meeting on Mathematics for Industry

6th to 8th June 2013

Faculdade de Ciências da Universidade do Porto
Rua do Campo Alegre, 687, 4169-007 Porto
Portugal
http://cmup.fc.up.pt/cmup/apmind/meeting2013/

The Portuguese Meetings on Mathematics for Industry will give sequence to Porto Meetings on the same topic. To view previous versions please see

http://cmup.fc.up.pt/cmup/apmind/meeting2013/

The purpose of this meeting is to focus the attention on the many and varied opportunities to promote applications of mathematics to industrial problems. Its major objectives are:

- Development and encouragement of industrial and academic collaboration, facilitating contacts between academic, industrial, business and finance users of mathematics
- Through "bridging the industrial/academic barrier" these meetings will provide opportunities to present successful collaborations and to elaborate elements such as technology transfer, differing vocabularies and goals, nurturing of contacts and resolution of issues.
- To attract undergraduate students to distinctive and relevant formation profiles, motivate them during their study, and advance their personal training in Mathematics and its Applications to Industry, Finance, etc.

The meeting will be focused on short courses, of three one-hour lectures each, given by invited distinguished researchers, which are supplemented by contributed short talks by other participants and posters of case studies. This edition is especially dedicated to two main themes:

- Optimization and Financial Mathematics
- Mathematical Epidemiology

Special participation of members of the Analytics & Decision Models Area, Millennium bcp/Banco Comercial Português, SA.

The meeting is promoted by APMInd (Portuguese Association of Mathematics for Industry: http://cmup.fc.up.pt/cmup/apmind/) as one of the activities of both GEMAC (Gabinete de Estatística, Modelação e Aplicações Computacionais: http://cmup.fc.up.pt/cmup/gemac/) and Master course in Mathematical Engineering (http://www.fc.up.pt/dmat/engmat/).

An Interview

with the Scientific Committee members of the Iberian Meeting on Numerical Semigroups — Vila Real 2012

by **Manuel Delgado** [CMUP and DM-FCUP, University of Porto] and
Pedro García Sánchez [Dep. Álgebra, University of Granada]

By the occasion of the Iberian Meeting on Numerical Semigroups -- Vila Real 2012, that held at the UTAD, from the 18th to the 20th of July, we posed some questions to the members of the Scientific Committee. These are Valentina Barucci, from the Università di Roma *La Sapienza*; José Carlos Rosales, from the Universidad de Granada; Ralf Fröberg, from the Stockholms Universitet; and Scott Chapman, from the Sam Houston State University.

Valentina Barucci most cited work (with over 70 cites) on numerical semigroups is her book on maximality properties on numerical semigroups with applications to one-dimensional analytically irreducible local domains, which has been for years a dictionary between these two mathematical objects. Valentina gave a talk in this meeting on differential operators on semigroup rings.

José Carlos Rosales wrote his thesis on numerical semigroups, and since then he wrote more than a hundred papers on this and related topics.

Ralf Fröberg most famous paper on numerical semigroups is *On numerical semigroups* that used numerical semigroups to solve problems on one-dimensional local rings. This paper has been cited over 65 times. In this meeting he gave a talk showing that homological tools can be used to give elegant answers to questions arising in the study of numerical semigroups.

Scott Chapman uses numerical semigroups in the study of non-unique factorization invariants, and also in the study of ideal theory on semigroup rings. He is currently editor in chief of the American Mathematical Monthly.

Can you tell us when did your interest on numerical semigroups started? And what was the motivation?

Valentina.— I started to be interested in numerical semigroups after meeting Ralf Fröberg, who proposed me some problems in this subject. The problems were very concrete and at the same time they implied some consequences for geometric objects, as monomial curves, so I felt attached soon.

José Carlos.— I started doing mathematical research by treating problems in this field. My doctoral thesis defended in Granada in 1991 was entitled Numerical Semigroups.

Ralf.— I became interested in numerical semigroups in the 1980's. Mostly by chance, we and two colleagues (Gottlieb, Häggkvist) started to discuss the subject and finally we wrote a paper. Then I met Valentina Barucci and we started to work on problems concerning numerical semigroup rings. This joint work continued for many years, and later included Marco D'Anna. Our interest comes from one-dimensional rings, which has a lot to do with numerical semigroups.

Scott.— My main doctoral thesis problem involved studying the ideal theory of particular semigroup rings. While the actual work was in Commutative Algebra, I was fascinated by the structure of numerical monoids. As I began to work more and more on factorization problems, I began applying the ideas and techniques of the theory of non-unique factorizations to numerical monoids and semigroups.

Would you recommend the study of numerical semigroups to start a research career?

José Carlos.— Over the years I have never abandoned the research on this topic. It is to me an exciting field in which many problems appear naturally and in which there are increasingly more researchers involved. This is the reason why I, undoubtedly, would recommend this field to someone interested in research in mathematics.

Scott.— Enthusiastically! Many of my publications involving numerical monoids and semigroups have been written with students who worked under my direction on research supported by the National Science Foundation. While problems in numerical semigroups become extremely difficult, the background for getting started is merely rational number theory. Students can quickly get basic results, but just as quickly learn how challenging mathematics can become.

Have you had students that successfully defended their PHD thesis in numerical semigroups?

José Carlos.— I have supervised 5 doctoral theses and in all of them the study of numerical semigroups has played an important role.

Ralf.— I have had two students, who partly wrote their thesis on numerical semigroups, and I also worked informally as supervisor to two guest students from Italy, mainly on numerical semigroups.

Scott.— My institution does not support a doctoral program. But, many of my past students who started their careers studying numerical monoids, later went on to complete Ph.D. degrees at very high quality American institutions. Moreover, they remain interested in the subject and have even in the past attended at least one of the meetings in the IMNS series. Examples are Paul Baginski (Ph.D. University of California at Berkeley) and Nathan Kaplan (Ph.D. expected this year from Harvard).

Do you plan to continue working on numerical semigroups? Would you like to mention some open question in the area that you feel like one of the most important?

José Carlos.— Absolutely. There are many issues to deepen and many open problems which, undoubtedly, attract the attention of many mathematicians. In this line I would highlight the Frobenius problem, the Wilf conjecture and the Bras-Amoros conjecture.

Scott.— Yes, I plan on continuing to work in this area. Here is an open problem that I think is particularly important. Let S be a numerical monoid and assume its elements are listed in increasing order as $S = s_1, s_2, s_3, \ldots$. In the paper Delta sets of numerical monoids are eventually periodic, (Aequationes Math. 77 (2009), 273–279) written by Chapman, Hoyer and Kaplan, it is shown the the sequence of delta sets $\Delta(s_1), \Delta(s_2), \Delta(s_3), \ldots$ is eventually periodic. Is the same true for the sequences $c(s_1), c(s_2), c(s_3), \ldots$ and $t(s_1), t(s_2), t(s_3), \ldots$ where $c(x)$ represents the catenary degree of x in S and $t(x)$ the tame degree of x in S?

Which is from your point of view the most important question on numerical semigroups that has been solved? What was its impact and relevance in other areas of Mathematics?

José Carlos.— The study of numerical semigroups is a classic theme. However, from the second half of the twentieth century suffered a major boost due to its applications in many and interesting fields such as algebraic geometry, coding theory, number theory and computer algebra.

Ralf.— I am not able to point at the most important theorem in numerical semigroup theory. I think that several connections with other subjects will be found, so perhaps it is too early to say what is most important.

Scott.— My interest in numerical semigroups centers in the study of non-unique factorizations. While there are significant papers concerning the structure of numerical semigroups, I will focus on their factorization properties in answering this question. I think the solution of the question cited above (about the eventual periodicity of the sequence of delta sets) is a very important result. The wealth of results on the factorization properties of numerical monoids has led to similar investigations in other types of monoids (such as block monoids, Diophantine monoids and congruence monoids) which might have otherwise never been completed.

Has the study of numerical semigroups some relevance in the mathematics developed in your country? Is it difficult to get funding for doing research in numerical semigroups?

José Carlos.— In my opinion the importance of this line of research is comparable to the most relevant lines of mathematical research. That is why we have not had difficulties in obtaining funds through national and regional research projects.

Ralf.— It is generally hard to get money for research in Sweden, not particularly for semigroups.

Scott.— Yes, it has particular relevance in Commutative Algebra and Algebraic Geometry. It is difficult in the United States to obtain funding for almost any pure mathematical subject. I have over the past 15 years obtained such funding on three different occasions from the National Science Foundation to run Research Experiences in Mathematics Programs for undergraduate students.

This is the third edition of the Iberian Meeting on Numerical Semigroups. The authors of this interview have been in charge of the organization of all the three editions, with very important collaborations in the second and third. Its success has exceeded by far the initial expectations for a meeting that was intended to gather together mathematicians from various areas where numerical semigroups appear in a rather informal way. Please

Membros da Comissão Científica, organizadores e organizadores locais do IMNS 2012. Da esquerda para a direita: Luís Roçadas, UTAD, Ralf Froberg, Stockholms Universitet, Scott Chapman, Sam Houston State University, Valentina Barrucci, Univ. di Roma *La Sapienza*, Paula Catarino, UTAD, P. A. García-Sánchez, Universidad de Granada, José Carlos Rosales, Universidad de Granada, André Oliveira, UTAD, M. Delgado, Universidade de Porto, Paulo Vasco, UTAD

give us your opinion on the usefulness or not of this kind of meetings for the development of an area, in particular in an area that constitutes a small intersection point of several others.

Valentina.—Numerical semigroups is apparently a narrow subject, but it gathers people from different areas.

Thus, the most interesting talks in the meeting were for me the talks where also other subjects, e.g. from commutative algebra or from code theory, appear.

It was also interesting and useful to meet personally some mathematicians who worked on similar problems than me and that I know only though their papers.

The meeting was also pleasant because there are not people that consider themselves big stars, as sometime happens, and there was a very nice cooperative atmosphere.

José Carlos.—These meetings are very useful since they encourage the grouping of mathematicians interested in the study and applications of numerical semigroups coming from all over the world. This provides a contact at first hand with the latest advances in this field. It also provides discussions between different researchers that could not happen otherwise.

Ralf.—It is very important to have a chance to meet people in this relatively narrow area. This gives possibilities to talk about problems, but still to get views from different angles.

Scott.—I think it is highly useful. Most of the participants at the IMNS meetings found numerical semigroups by working in some other area. In my case, it was Commutative Algebra. In other cases, it was Computer Science, Graph Theory, Algebraic Geometry, …. The list is almost endless. I am not so sure that the intersection mentioned above is so small. In fact, I think it has grown drastically over the past 10 years and I believe that attendance at the next IMNS meetings will exceed that of any of the first three editions of this congress.

The multivariate extremal index and tail dependence

by Helena Ferreira*

Abstract.—If we obtain a tail dependence coefficient of the common distribution of the vectors in a multivariate stationary sequence then we do not have necessarily the correspondent coefficient of the limiting multivariate extreme value model. In opposition to sequences of independent and identically distributed random vectors, the local clustering of extremes allowed by stationarity can increase or decrease the tail dependence.

The temporal dependence at extreme levels can be summarized by the function multivariate extremal index and its effect in the tail dependence is well illustrated with Multivariate Maxima of Moving Maxima processes.

Keywords.—multivariate moving maxima, multivariate extremal index, tail dependence, multivariate extreme value distribution.

1. Introduction

For a random vector $\mathbf{X} = (X_1, \dots, X_d)$ with continuous marginal distributions $F_1, \dots, F_d$ and copula C, let the bivariate (upper) tail dependence coefficients $\lambda_{jj'}^{(\mathbf{X})} \equiv \lambda_{jj'}^{(C)}$ be defined by

$$\lim_{u\uparrow 1} P\left(F_j(X_j) > u | F_{j'}(X_{j'}) > u\right), \quad (1)$$

for $1 \le j < j' \le d$. Tail dependence coefficients measure the probability of extreme values occurring for one random variable given that another assumes an extreme value too. Positive values correspond to tail dependence and null values mean tail independence. These coefficients can be defined via copulas and it holds that

$$\lambda_{jj'}^{(C)} = 2 - \lim_{u\uparrow 1} \frac{\ln C_{jj'}(u,u)}{\ln u}, \quad (2)$$

where $C_{jj'}$ is the copula of the sub-vector $(X_j, X_{j'})$ ([4], [7]).

Let F be a multivariate distribution function with continuous marginals, which is in the domain of attraction of a Multivariate Extreme Value (henceforth MEV) distribution $\hat{H}$ with standard Fréchet margins, that is, $F^n(nx_1, \dots, nx_d) \underset{n\to\infty}{\longrightarrow} \hat{H}(x_1, \dots, x_d)$ with marginal distributions $\hat{H}(x_j) = \exp\left(-x_j^{-1}\right)$, $x_j > 0$, $j = 1, \dots, d$. It is known that any bivariate tail dependence coefficient of F is the same as the corresponding coefficient of $\hat{H}$ ([8]).

Let $\{\mathbf{Y}_n\}_{n\ge 1}$ be a multivariate stationary sequence such that $F_{\mathbf{Y}_n} = F$ and $\mathbf{M}_n = (M_{n1}, \dots, M_{nd})$ is the vector of componentwise maxima from $\mathbf{Y}_1, \dots, \mathbf{Y}_n$. If $\lim_{n\to\infty} P(M_{n1} \le nx_1, \dots, M_{nd} \le nx_d) = H(x_1, \dots, x_d)$, for some MEV distribution H, one question that naturally arises is Does dependence across the sequence affect the bivariate tail dependence of the limiting MEV? In other words, we would like to know the relation between the tail dependence coefficients of the limiting MEV H and the limiting MEV

$$\hat{H}(x_1, \dots, x_d) = \lim_{n\to\infty} P\left(\hat{M}_{n1} \le nx_1, \dots, \hat{M}_{nd} \le nx_d\right) = \lim_{n\to\infty} F^n(nx_1, \dots, nx_d),$$

where $\hat{\mathbf{M}}_n = (\hat{M}_{n1}, \dots, \hat{M}_{nd})$ is the vector of pointwise maxima for a sequence of i.i.d. random vectors $\{\hat{\mathbf{Y}}_n\}_n \ge 1$ associated to $\{\mathbf{Y}_n\}_n \ge 1$, that is, such that, $F_{\hat{\mathbf{Y}}_n} = F_{\mathbf{Y}_n} = F$.

Our main purpose is to compare the bivariate tail dependence coefficients for the margins of the two Multivariate Extreme Value distributions H and $\hat{H}$ through the function multivariate extremal index ([6]), which resumes temporal dependence in $\{\mathbf{Y}_n\}_{n\ge 1}$.

We recall that the d–dimensional stationary sequence $\{\mathbf{Y}_n\}_{n\ge 1}$ is said to have a multivariate extremal index $\theta(\boldsymbol{\tau}) \in [0,1]$, $\boldsymbol{\tau} = (\tau_1, \dots, \tau_d) \in \mathbb{R}_+^d$, if for each $\boldsymbol{\tau}$ in $\mathbb{R}_+^d$,

* Department of Mathematics — University of Beira Interior, Covilhã

there exists levels $\mathbf{u}_n^{(\tau)} = (u_{n1}^{(\tau_1)}, \dots, u_{nd}^{(\tau_d)})$, $n \geq 1$, satisfying

$$nP(Y_{1j} > u_{nj}^{(\tau_j)}) \underset{n\to\infty}{\longrightarrow} \tau_j,\ j = 1, \dots, d,$$

$$P(\hat{\mathbf{M}}_n \leq u_n^{(\tau)}) \underset{n\to\infty}{\longrightarrow} \hat{\gamma}(\boldsymbol{\tau}) > 0$$

and

$$P(\mathbf{M}_n \leq \mathbf{u}_n^{(\tau)}) \underset{n\to\infty}{\longrightarrow} \hat{\gamma}(\boldsymbol{\tau})^{\theta(\tau)}.$$

The multivariate extremal index, although dependent of $\boldsymbol{\tau}$, is an homogeneous function of order zero and if it exists for $\{\mathbf{Y}_n\}_{n\geq 1}$ then any sequence of sub-vectors $\{(\mathbf{Y}_n)_A\}_{n\geq 1}$ with indices in $A \subset \{1, \dots, d\}$ has multivariate extremal index

$$\theta_A(\boldsymbol{\tau}_A) = \lim_{\substack{\tau_i\to 0^+\\ i\notin A}} \theta(\tau_1, \dots, \tau_d),\ \boldsymbol{\tau}_A \in \mathbb{R}_+^{|A|}.$$

Moreover, for each marginal sequence $\{Y_{n,j}\}_{n\geq 1}$, the extremal index $\theta_{\{j\}} \equiv \theta_j$ is constant.

Under suitable long range dependence conditions for the stationary sequence $\{\mathbf{Y}_n\}_{n\geq 1}$, like strong-mixing condition, and for $r_n = o(n)$ we have

$$\frac{1}{\theta(\boldsymbol{\tau})} = \lim_{n\to+\infty} E\left(\sum_{k=1}^{r_n} \mathbb{I}_{\{\mathbf{Y}_k \nleq \mathbf{u}_n^{(\tau)}\}} \middle| \sum_{k=1}^{r_n} \mathbb{I}_{\{\mathbf{Y}_k \nleq \mathbf{u}_n^{(\tau)}\}} > 0\right),$$

where $\mathbb{I}_A$ denotes the indicator function. That is, the extremal index is the reciprocal of the limiting mean of the cluster size of exceedances of $\mathbf{u}_n^{(\tau)}$.

As motivation, we first compare in the next section the bivariate tail dependence coefficients of $\hat{H}$ and H arising in two Multivariate Maxima Moving Maxima (henceforth M4) processes ([12]), where we can compute directly these coefficients. Next we present the main result with a corollary on the M4 processes.

Since bivariate tail dependence for MEV distributions is related with the extremal coefficients ([13], [11]), we translate the main result in terms of these coefficients.

Finally, in the last section we discuss the multivariate generalizations of the results.

2. M4 examples

Let $\{Z_{l,n}\}_{l\geq 1, -\infty<n<\infty}$ be an array of independent random vectors with standard Fréchet margins. A multivariate maxima of moving maxima process is defined by

$$Y_{n,j} = \bigvee_{l\geq 1} \bigvee_{-\infty<k<\infty} \alpha_{lkj} Z_{l,n-k},\ j = 1, \dots, d, \qquad (3)$$

$n \geq 1$, where $\{\alpha_{lkj}, l \geq 1, -\infty < k < \infty, 1 \leq j \leq d\}$, are nonnegative constants satisfying

$$\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \alpha_{lkj} = 1 \quad \text{for } j = 1, \dots, d.$$

Extremal behaviour of these processes was developed in [12], where it is proved that the sequence $\{\mathbf{Y}_n\}_{n\geq 1}$ has multivariate extremal index

$$\theta(\tau_1, \dots, \tau_d) = \frac{\sum_{l=1}^{\infty} \bigvee_{-\infty\leq k\leq+\infty} \bigvee_{j=1,\dots,d} \alpha_{lkj}\tau_j}{\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \bigvee_{j=1,\dots,d} \alpha_{lkj}\tau_j}.$$

and the extremal index of $\{Y_{nj}\}_{n\geq 1}$ is

$$\theta_j = \sum_{l=1}^{\infty} \bigvee_{-\infty<k<\infty} \alpha_{lkj},\ j = 1, \dots, d.$$

The M4 class of processes, which are very flexible models for temporally dependent processes, plays a remarkable role in the multivariate extreme value theory since the multivariate extremal index of a stationary max-stable sequence $\{\mathbf{Y}_n\}_{n\geq 1}$ may be approximated uniformly by the multivariate extremal index of an M4 sequence ([1]).

The common copula $C_{\mathbf{Y}}$ of $\mathbf{Y}_n = (Y_{n,1}, \dots, Y_{n,d})$ is defined by

$$C_{\mathbf{Y}}(u_1, \dots, u_d) = \prod_{l=1}^{\infty} \prod_{k=-\infty}^{\infty} \left(u_1^{\alpha_{lk1}} \wedge \dots \wedge u_d^{\alpha_{lkd}}\right),$$

$u_j \in [0, 1]$, $j = 1, \dots, d$. The dependence across the d dimensions is regulated by the structure of the signatures a_{lkj} in each l–th moving pattern. We will consider two examples with a finite number l of moving patterns and finite range $k_1 \leq k \leq k_2$ for its signatures.

Example 1.—We first consider a M4 process with one moving pattern and finite range $-1 \leq k \leq 1$ for the sequence dependence, defined as follows. For $n \geq 1$, let

$$\begin{cases} Y_{n,1} = \frac{1}{8}Z_{1,n-1} \vee \frac{1}{8}Z_{1,n} \vee \frac{6}{8}Z_{1,n+1} \\ Y_{n,2} = \frac{2}{8}Z_{1,n-1} \vee \frac{1}{8}Z_{1,n} \vee \frac{5}{8}Z_{1,n+1} \end{cases}$$

We have in this case,

$$C_{\mathbf{Y}}(u_1, u_2) \equiv C_{\mathbf{Y}_n}(u_1, u_2) = \left(u_1^{1/8} \wedge u_2^{2/8}\right)\left(u_1^{1/8} \wedge u_2^{1/8}\right)\left(u_1^{6/8} \wedge u_2^{5/8}\right)$$

and

$$\lambda^{(C_{\mathbf{Y}})} = \lambda^{(\hat{C})} = 2 - \left(\frac{2}{8} + \frac{1}{8} + \frac{6}{8}\right) = \frac{7}{8},$$

where $\hat{C}$ denotes the copula of $\hat{H}$. Otherwise

$$H(x_1, x_2) = \exp\left(-\left(\frac{6x_1^{-1}}{8} \vee \frac{5x_2^{-1}}{8}\right)\right).$$

Therefore $C(u_1, u_2) = u_1 \wedge u_2$ and $\lambda^{(C)} = 1 > \lambda^{(\hat{C})}$.

Example 2.—We will now consider a modification in the above example through the introduction of one more pattern. Let, for each $n \geq 1$,

$$\begin{cases} Y_{n,1} = \frac{1}{8}Z_{1,n} \vee \frac{6}{8}Z_{1,n+1} \vee \frac{1}{8}Z_{2,n} \\ Y_{n,2} = \frac{1}{8}Z_{1,n} \vee \frac{5}{8}Z_{1,n+1} \vee \frac{2}{8}Z_{2,n} \end{cases}.$$

We have the same $C_{\mathbf{Y}}$ and $\lambda^{(\hat{C})} = 7/8$ as in the previous example, but here

$$H(x_1, x_2) = \exp\left(-\left(\frac{6x_1^{-1}}{8} \vee \frac{5x_2^{-1}}{8}\right)\right)\exp\left(-\left(\frac{x_1^{-1}}{8} \vee \frac{2x_2^{-1}}{8}\right)\right)$$

and therefore

$$C(u_1, u_2) = \left(u_1^{6/7} \wedge u_2^{5/7}\right)\left(u_1^{1/7} \wedge u_2^{2/7}\right).$$

Then $\lambda^{(C)} = 2 - (6/7 + 2/7) = 6/7 < \lambda^{(\hat{C})}$.

These examples show that the dependence structure of the sequence can increase or decrease the tail dependence coefficients in the limiting MEV model. In the next proposition we will quantify such variation through the function multivariate extremal index.

3. Main result

In the this section we will relate $\lambda_{jj'}^{(C)}$ with $\lambda_{jj'}^{(\hat{C})}$, where C is the copula of the MEV distribution H and $\hat{C}$ corresponds to the limiting MEV distribution for the associated i.i.d. sequence. We first remark that it follows from the definition of the multivariate extremal index

$$\hat{H}\left(\frac{1}{\tau_j}, \frac{1}{\tau_{j'}}\right) = \hat{C}_{jj'}\left(e^{-\tau_j}, e^{-\tau_{j'}}\right),$$

and

$$H\left(\frac{1}{\tau_j}, \frac{1}{\tau_{j'}}\right) = C_{jj'}\left(e^{-\tau_j\theta_j}, e^{-\tau_{j'}\theta_{j'}}\right).$$

Then

$$C_{jj'}\left(e^{-\tau_j\theta_j}, e^{-\tau_{j'}\theta_{j'}}\right) = \left(\hat{C}_{jj'}\left(e^{-\tau_j}, e^{-\tau_{j'}}\right)\right)^{\theta(\tau_j, \tau_{j'})},$$

where $\theta(\tau_j, \tau_{j'})$ is the bivariate extremal index of $\{(Y_{nj}, Y_{nj'})\}_{n\geq 1}$. Therefore, for each $(u_j, u_{j'}) \in [0, 1]^2$, we have

$$C_{jj'}(u_j, u_{j'}) = \left(\hat{C}_{jj'}\left(u_j^{1/\theta_j}, u_{j'}^{1/\theta_{j'}}\right)\right)^{\theta\left(-\frac{\ln u_j}{\theta_j}, -\frac{\ln u_{j'}}{\theta_{j'}}\right)}. \qquad (4)$$

This relation enables us to compare the tail dependence coefficients $\lambda_{jj'}^{(C)}$ with $\lambda_{jj'}^{(\hat{C})}$ through the function $\theta(\tau_j, \tau_{j'})$.

Proposition 3.1.—If C and $\hat{C}$ satisfy (4) then, for each $1 \leq j < j' \leq d$, it holds that

(a) $\lambda_{jj'}^{(C)} = 2 + \theta\left(\frac{1}{\theta_j}, \frac{1}{\theta_{j'}}\right) \ln \hat{C}_{jj'}\left(e^{-1/\theta_j}, e^{-1/\theta_{j'}}\right)$,

(b) $\lambda_{jj'}^{(C)} = \lambda_{jj'}^{(\hat{C})} + \ln \dfrac{\hat{C}_{jj'}\left(e^{-\theta\left(\frac{1}{\theta_j}, \frac{1}{\theta_{j'}}\right)/\theta_j}, e^{-\theta\left(\frac{1}{\theta_j}, \frac{1}{\theta_{j'}}\right)/\theta_{j'}}\right)}{\hat{C}_{jj'}\left(e^{-1}, e^{-1}\right)}$.

Proof.—From the spectral measure representation of $\hat{H}$ ([2]), the copula $\hat{C}$ can be written as follows:

$$\hat{C}(u_1, \dots, u_d) = \exp\left(-\int_{\mathcal{S}_d} \bigvee_{j=1}^{d} w_j(-\ln u_j)\, d\hat{W}(\mathbf{w})\right),$$

where $\hat{W}$ is a finite measure on the unit shere $\mathcal{S}_d$ of $\mathbb{R}_+^d$ satisfying $\int_{\mathcal{S}_d} w_j\, d\hat{W}(\mathbf{w}) = 1$, $j = 1, \dots, d$. Then

$$\lambda_{jj'}^{(\hat{C})} = 2 - \lim_{u\uparrow 1} \frac{\ln \hat{C}_{jj'}(u, u)}{\ln u} = 2 - \int_{\mathcal{S}_d} (w_j \vee w_{j'})\, d\hat{W}(\mathbf{w}) = 2 + \ln \hat{C}_{jj'}(e^{-1}, e^{-1}). \qquad (5)$$

By using (4) and the homogeneity of order zero of the multivariate extremal index, it follows that

$$\lambda_{jj'}^{(C)} = 2 - \lim_{u\uparrow 1} \frac{\ln C_{jj'}(u, u)}{\ln u} =$$

$$-\theta\left(\frac{1}{\theta_j}, \frac{1}{\theta_{j'}}\right) \lim_{u\uparrow 1} \frac{\int_{\mathcal{S}_d}\left(\frac{-\ln u w_j}{\theta_j} \vee \frac{-\ln u w_{j'}}{\theta_{j'}}\right) d\hat{W}(\mathbf{w})}{-\ln u} =$$

$$-\theta\left(\frac{1}{\theta_j}, \frac{1}{\theta_{j'}}\right) \int_{\mathcal{S}_d}\left(\frac{w_j}{\theta_j} \vee \frac{w_{j'}}{\theta_{j'}}\right) d\hat{W}(\mathbf{w}) =$$

$$2 + \theta\left(\frac{1}{\theta_j}, \frac{1}{\theta_{j'}}\right) \ln \hat{C}_{jj'}\left(e^{-1/\theta_j}, e^{-1/\theta_{j'}}\right). \qquad (6)$$

To obtain the second statement we combine (a) and the max-stability of $\hat{C}_{jj'}$. ⊣

The previous result is valid to any MEV copulas C and $\hat{C}$ copulas related by a multivariate extremal index $\theta(\tau_1, \dots, \tau_d)$, that is, satisfying the relation

$$C(u_1, \dots, u_d) = \left(\hat{C}\left(u_1^{1/\theta_1}, \dots, u_d^{1/\theta_d}\right)\right)^{\theta\left(-\frac{\ln u_1}{\theta_1}, \dots, -\frac{\ln u_d}{\theta_d}\right)}, \qquad (7)$$

$(u_1, \dots, u_d) \in [0, 1]^d$, which leads to (4).

For the particular case of M4 processes it is known ([3]) that

$$\lambda_{jj'}^{(\hat{C})} = 2 - \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} (\alpha_{lkj} \vee \alpha_{lkj'}),$$

which is in general greater than zero. We add now, as a corollary of the previous proposition, the expression of $\lambda_{jj'}^{(C)}$ for these processes.

COROLLARY 3.2.—Let $\{\mathbf{Y}_n\}_{n\geq 1}$ be a M4 process defined as in (2). Then, for any $1 \leq j < j' \leq d$, it holds that

$$\text{(a)}\ \lambda_{jj'}^{(C)} = 2 - \sum_{l=1}^{\infty}\left(\frac{\bigvee_{-\infty\leq k\leq+\infty} \alpha_{lkj}}{\theta_j} \vee \frac{\bigvee_{-\infty\leq k\leq+\infty} \alpha_{lkj'}}{\theta_{j'}}\right),$$

(b) if $l = 1$ then $\lambda_{jj'}^{(C)} = 1$,

(c) $\lambda_{jj'}^{(C)} > \lambda_{jj'}^{(\hat{C})}$ if and only if

$$\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}\left(\alpha_{lkj} \vee \alpha_{lkj'}\right) >$$

$$\sum_{l=1}^{\infty}\left(\bigvee_{-\infty\leq k\leq+\infty} \alpha_{lkj}/\theta_j \vee \bigvee_{-\infty\leq k\leq+\infty} \alpha_{lkj'}/\theta_{j'}\right).$$

We can apply the previous formula in (a) and find the same results in the previous examples.

As we have seen, it is easy to construct examples for which $\lambda_{jj'}^{(C)} > \lambda_{jj'}^{(\hat{C})}$ or $\lambda_{jj'}^{(C)} < \lambda_{jj'}^{(\hat{C})}$. However, if $\lambda_{jj'}^{(\hat{C})} = 1$ then $\lambda_{jj'}^{(C)} = 1$, as a consequence of the Proposition 3.1 and the Corollary 2 in [1], which states that $\lambda_{jj'}^{(\hat{C})} = 1$ leads to $\theta(\tau_j, \tau_{j'}) = \theta_j \wedge \theta_{j'}$. In fact, as a consequence of this result, we will have in (a) of the Proposition 3.1.,

$$\lambda_{jj'}^{(C)} = 2 - \frac{\theta(\frac{1}{\theta_j}, \frac{1}{\theta_{j'}})}{\theta_j \wedge \theta_{j'}} = 2 - \frac{\theta_j \wedge \theta_{j'}}{\theta_j \wedge \theta_{j'}} = 2 - 1 = 1.$$

Another popular dependence coefficient, which is closely related to the multivariate extemal index, is the extremal coefficient. It is a summary coefficient for the extremal dependence introduced by Tiago de Oliveira ([13]) for bivariate extreme value distributions and extended to MEV distributions in [11]. Let the extremal coefficient of the MEV copula C be the constant ϵ_C such that $C(u, \ldots, u) = u^{\epsilon_C}$, for all $u \in [0, 1]$.

Since

$$-\log C(u_1, \ldots, u_d) =$$

$$\theta\left(-\frac{\ln u_1}{\theta_1}, \ldots, -\frac{\ln u_d}{\theta_d}\right)\int_{\mathscr{S}_d}\bigvee_{j=1}^{d}\frac{-\ln u_j w_j}{\theta_j}\, d\hat{W}(\mathbf{w})\}, \quad (8)$$

we get the following relation between the extremal coefficient of C, the copula $\hat{C}$ and the multivariate extremal index.

PROPOSITION 3.3.—If C and $\hat{C}$ satisfy (7) then

$$\epsilon_C = -\theta\left(-\frac{1}{\theta_1}, \ldots, -\frac{1}{\theta_d}\right)\ln\hat{C}\left(e^{-1/\theta_1}, \ldots, e^{-1/\theta_d}\right).$$

This result extends in a natural way the known relation

$$\epsilon_{\hat{C}} = -\ln\hat{C}\,(e^{-1}, \ldots, e^{-1})$$

and enables to see the Proposition 3.1-(a) as an extension of the classical result $\lambda = 2 - \epsilon$ ([10]).

4. MULTIVARIATE TAIL DEPENDENCE COEFFICIENTS

How to characterise the strength of extremal dependence with respect to a particular subset of random variables of $\mathbf{X}$? One can use conditional orthant tail probabilities of $\mathbf{X}$ given that the components with indices in the subset J are extreme. The tail dependence of bivariate copulas can be extended as done in [9] and [5].

For $\varnothing \neq J \subset D = \{1, \ldots, d\}$, let $\lambda_J^{(\mathbf{X})} \equiv \lambda_J^{(C)} =$

$$\lim_{u\uparrow 1} P\left(\bigcap_{j\notin J} F_j(X_j) > u \,|\, \bigcap_{j\in J} F_j(X_j) > u\right). \quad (9)$$

If for some $\varnothing \neq J \subset \{1, \ldots, d\}$ the coefficient $\lambda_J^{(C)}$ exists and is positive then we say that $\mathbf{X}$ is (upper) orthant tail dependent. The relation (2) between the tail dependence coefficient and the bivariate copula can also be generalized by

$$\lambda_J^{(C)} = \lim_{u\uparrow 1}\frac{\sum_{\varnothing\neq S\subset D}(-1)^{|S|-1}\ln C_S(\mathbf{u}_S)}{\sum_{\varnothing\neq S\subset J}(-1)^{|S|-1}\ln C_S(\mathbf{u}_S)}, \quad (10)$$

where C_S denotes de copula of $\mathbf{X}_S$ and $\mathbf{u}_S$ the $|S|$-dimensional vector $(u, \ldots, u)$.

If $F^n(nx_1, \ldots, nx_d) \to \hat{H}(x_1, \ldots, x_d)$ then any positive tail dependence coefficient $\lambda_J^{(F)}$ of F is the same as the corresponding tail dependence coefficient $\lambda_J^{(\hat{C})}$ of the limiting MEV $\hat{H}$ ([5]). In the case of $\theta(\boldsymbol{\tau}) \neq 1$, the limiting MEV model does not preserve orthant tail dependence coefficients.

Let $\theta_S^* \equiv \theta_S((1/\theta_1, \ldots, 1/\theta_d)_S)$denotes the multivariate extremal index of the sequence of sub-vectors $\{(\mathbf{Y}_n)_S\}_{n\geq 1}$ in the sub-vector of $\theta(1/\theta_1, \ldots, 1/\theta_d)$ with components in S.

By using (8) and (10), we can compute $\lambda_J^{(C)}$ from the copula $\hat{C}$ as follows.

PROPOSITION 4.1.—If C and $\hat{C}$ satisfy (7) then

$$\lambda_J^{(C)} = \frac{\sum_{\varnothing\neq S\subset D}(-1)^{|S|-1}\theta_S^*\ln\hat{C}_S\left(\left(e^{-1/\theta_1}, \ldots, e^{-1/\theta_d}\right)_S\right)}{\sum_{\varnothing\neq S\subset J}(-1)^{|S|-1}\theta_S^*\ln\hat{C}_S\left(\left(e^{-1/\theta_1}, \ldots, e^{-1/\theta_d}\right)_S\right)},$$

provided the ratio is defined.

We will illustrate the above proposition with the M4 processes.

EXAMPLE 3.—Let $\{\mathbf{Y}_n\}_n \geq 1$ defined by

$$\begin{cases} Y_{n,1} = \frac{1}{8}Z_{1,n-1} \vee \frac{3}{8}Z_{1,n} \vee \frac{4}{8}Z_{1,n+1} \\ Y_{n,2} = \frac{2}{8}Z_{1,n-1} \vee \frac{2}{8}Z_{1,n} \vee \frac{4}{8}Z_{1,n+1} \\ Y_{n,3} = \frac{4}{8}Z_{1,n-1} \vee \frac{2}{8}Z_{1,n} \vee \frac{2}{8}Z_{1,n+1} \end{cases}$$

We have

$$C_{\mathbf{Y}}(u_1, u_2, u_3) = \left(u_1^{1/8} \wedge u_2^{2/8} \wedge u_3^{4/8}\right)$$
$$\left(u_1^{3/8} \wedge u_2^{2/8} \wedge u_3^{2/8}\right)\left(u_1^{4/8} \wedge u_2^{4/8} \wedge u_3^{2/8}\right),$$

$$\theta\left(\frac{1}{\theta_1}, \frac{1}{\theta_2}, \frac{1}{\theta_3}\right) = \frac{4}{11},\ \theta_{\{1,2\}}\left(\frac{1}{\theta_1}, \frac{1}{\theta_2}\right) = \frac{4}{9},$$

$$\theta_{\{1,3\}}\left(\frac{1}{\theta_1}, \frac{1}{\theta_3}\right) = \frac{4}{11} \text{ and } \theta_{\{2,3\}}\left(\frac{1}{\theta_2}, \frac{1}{\theta_3}\right) = \frac{4}{10}.$$

Therefore

$$\lambda_{\{1,2\}}^{(C_Y)} = \lambda_{\{1,2\}}^{(\hat{C})} = \frac{-3 + \frac{9}{8} + \frac{11}{8} + \frac{10}{8} - \frac{10}{8}}{-2 + \frac{9}{8}} = \frac{4}{7}$$

and

$$\lambda_{\{1,2\}}^{(C)} = \frac{-3 + 1 + 1 + 1 - \frac{10}{11}}{-2 + 1} = \frac{10}{11} > \lambda_{\{1,2\}}^{(\hat{C})}.$$

EXEMPLO 4.—Let us consider now

$$\begin{cases} Y_{n,1} = \frac{1}{8}Z_{1,n} \vee \frac{6}{8}Z_{1,n+1} \vee \frac{1}{8}Z_{2,n} \\ Y_{n,2} = \frac{1}{8}Z_{1,n} \vee \frac{5}{8}Z_{1,n+1} \vee \frac{2}{8}Z_{2,n} \\ Y_{n,3} = \frac{1}{8}Z_{1,n} \vee \frac{7}{8}Z_{1,n+1} \end{cases}$$

We have

$$C_Y(u_1, u_2, u_3) = \left(u_1^{1/8} \wedge u_2^{1/8} \wedge u_3^{1/8}\right)$$
$$\left(u_1^{6/8} \wedge u_2^{5/8} \wedge u_3^{7/8}\right)\left(u_1^{1/8} \wedge u_2^{2/8}\right),$$

$$\theta\left(\frac{1}{\theta_1}, \frac{1}{\theta_2}, \frac{1}{\theta_3}\right) = \frac{9}{10},\ \theta_{\{1,2\}}\left(\frac{1}{\theta_1}, \frac{1}{\theta_2}\right) = \frac{8}{9},$$

$$\theta_{\{1,3\}}\left(\frac{1}{\theta_1}, \frac{1}{\theta_3}\right) = 1 \text{ and } \theta_{\{2,3\}}\left(\frac{1}{\theta_2}, \frac{1}{\theta_3}\right) = \frac{9}{10}.$$

Therefore

$$\lambda_{\{2,3\}}^{(C_\mathbf{Y})} = \lambda_{\{2,3\}}^{(\hat{C})} = \frac{-3 + \frac{9}{8} + \frac{9}{8} + \frac{10}{8} - \frac{10}{8}}{-2 + \frac{10}{8}} = 1$$

and

$$\lambda_{\{2,3\}}^{(C)} = \frac{-3 + \frac{8}{7} + \frac{9}{7} + \frac{9}{7} - \frac{9}{7}}{-2 + \frac{9}{7}} = \frac{4}{5} < 1 = \lambda_{\{2,3\}}^{(\hat{C})}.$$

Examples show that dependence across the sequence can increase or decrease the tail dependence coefficients and $\lambda_J^{(\hat{C})} = 1$ does not imply that $\lambda_J^{(C)} = 1$ with J consisting of more than one index. Finally, Proposition 3.1 can be extended as

$$\lambda_{\{i\}}^{(C)} = \sum_{\varnothing\neq S\subset D}(-1)^{|S|-1}\theta_S^*\ln\hat{C}_S\left(\left(e^{-1/\theta_1}, \ldots, e^{-1/\theta_d}\right)_S\right) =$$

$$\lambda_{\{i\}}^{(\hat{C})} + \sum_{\varnothing\neq S\subset D}(-1)^{|S|-1}\ln\frac{\hat{C}_S\left(\left(e^{-\theta_S^*/\theta_1}, \ldots, e^{-\theta_S^*/\theta_d}\right)_S\right)}{\hat{C}_S\left(\left(e^{-1}, \ldots, e^{-1}\right)_S\right)},$$

and a theoretical comparison between $\lambda_J^{(C)}$ and $\lambda_J^{(\hat{C})}$ follows from the relation

$$\lambda_J^{(C)} = \frac{\lambda_{\{i\}}^{(C)}}{\lambda_{\{i\}}^{(C_J)}},\ \lambda_{\{i\}}^{(C_J)} \neq 0.$$

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Numerical semigroups problem list

by M. Delgado*, P. A. García-Sánchez** and J. C. Rosales***

1. Notable elements and first problems

A numerical semigroup is a subset of $\mathbb{N}$ (here $\mathbb{N}$ denotes the set of nonnegative integers) that is closed under addition, contains the zero element, and its complement in $\mathbb{N}$ is finite.

If A is a nonempty subset of $\mathbb{N}$, we denote by $\langle A\rangle$ the submonoid of $\mathbb{N}$ generated by A, that is,

$$\langle A\rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N}, \lambda_i \in \mathbb{N}, a_i \in A\}.$$

It is well known (see for instance [41, 45]) that $\langle A\rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$.

If S is a numerical semigroup and $S = \langle A\rangle$ for some $A \subseteq S$, then we say that A is a system of generators of S, or that A generates S. Moreover, A is a minimal system of generators of S if no proper subset of A generates S. In [45] it is shown that every numerical semigroup admits a unique minimal system of generators, and it has finitely many elements.

Let S be a numerical semigroup and let $\{n_1 < n_2 < \cdots < n_e\}$ be its minimal system of generators. The integers n_1 and e are known as the multiplicity and embedding dimension of S, and we will refer to them by using $\mathrm{m}(S)$ and $\mathrm{e}(S)$, respectively. This notation might seem amazing, but it is not so if one takes into account that there exists a large list of manuscripts devoted to the study of analytically irreducible one-dimensional local domains via their value semigroups, which are numerical semigroups. The invariants we just introduced, together with others that will show up later in this work, have an interpretation in that context, and this is why they have been named in this way. Along this line, [3] is a good reference for the translation for the terminology used in the Theory of Numerical Semigroups and Algebraic Geometry.

Frobenius (1849–1917) during his lectures proposed the problem of giving a formula for the greatest integer that is not representable as a linear combination, with nonnegative integer coefficients, of a fixed set of integers with greatest common divisor equal to 1. He also raised the question of determining how many positive integers do not admit such a representation. With our terminology, the first problem is equivalent to that of finding a formula in terms of the generators of a numerical semigroup S of the greatest integer not belonging to S (recall that its complement in $\mathbb{N}$ is finite). This number is thus known in the literature as the Frobenius number of S, and we will denote it by $\mathrm{F}(S)$. The elements of $\mathrm{H}(S) = \mathbb{N} \setminus S$ are called gaps of S. Therefore the second problem consists in determining the cardinality of $\mathrm{H}(S)$, sometimes known as genus of S ([25]) or degree of singularity of S ([3]).

In [60] Sylvester solves the just quoted problems of Frobenius for embedding dimension two. For semigroups with embedding dimension greater than or equal to three these problems remain open. The current state of the problem is quite well collected in [30].

Let S be a numerical semigroup. Following the terminology introduced in [39] an integer x is said to be a pseudo-Frobenius number of S if $x \notin S$ and $x + S \setminus \{0\} \subseteq S$. We will denote by $\mathrm{PF}(S)$ the set of pseudo-Frobenius numbers of S. The cardinality of $\mathrm{PF}(S)$ is called the type of S(see [3]) and we will denote it by $\mathrm{t}(S)$. It is proved in [18] that if $\mathrm{e}(S) = 2$, then $\mathrm{t}(S) = 1$, and if $\mathrm{e}(S) = 3$, then $\mathrm{t}(S) \in \{1, 2\}$. It is also shown that if $\mathrm{e}(S) \geq 4$, then $\mathrm{t}(S)$ can be arbitrarily large, $\mathrm{t}(S) \leq \mathrm{m}(S) - 1$ and that $(\mathrm{t}(S) + 1)\mathrm{g}(S) \leq \mathrm{t}(s)(\mathrm{F}(S) + 1)$. This is the starting point of a new line of research that consists in trying to determine the type of a numerical semigroup, once other invariants like multiplicity, embedding dimension, genus or Frobenius number are fixed.

Wilf in [66] conjectures that if S is a numerical semigroup, then $\mathrm{e}(S)\mathrm{g}(S) \leq (\mathrm{e}(S) - 1)(\mathrm{F}(S) + 1)$. Some fami-

* CMUP and DM-FCUP, Univ. do Porto, Portugal, mdelgado@fc.up.pt
** Dep. de Álgebra, Univ. de Granada, España, pedro@ugr.es
*** Dep. de Álgebra, Univ. de Granada, España, jrosales@ugr.es

lies of numerical semigroups for which it is known that the conjecture is true are collected in [16]. Other such families can be seen in [23,59]. The general case remains open.

Bras-Amorós computes in [5] the number of numerical semigroups with genus $g \in \{0, \dots, 50\}$, and conjectures that the growth is similar to that of Fibonacci's sequence. However it has not been proved yet that there are more semigroups of genus g than of genus $g+1$. Several attempts already appear in the literature. Kaplan [23] uses an approach that involves counting the semigroups by genus and multiplicity. He poses many related conjectures which could be taken literally and be proposed here as problems. We suggest them to the reader. A different approach, dealing with the asymptoptical behavior of the sequence of the number of numerical semigroups by genus, has been followed by Zhao [69]. Some progress has been achieved by Zhai [68], but many questions remain open.

2. Proportionally modular semigroups

Following the terminology introduced in [52], a proportionally modular Diophantine inequality is an expression of the form $ax \bmod b \le cx$, with a, b and c positive integers. The integers a, b and c are called the factor, the modulus and the proportion of the inequality, respectively. The set $S(a,b,c)$ of solutions of the above inequality is a numerical semigroup. We say that a numerical semigroup is proportionally modular if it is the set of solutions of some proportionally modular Diophantine inequality.

Given a nonempty subset A of $\mathbb{Q}_0^+$, we denote by $\langle A \rangle$ the submonoid of $(\mathbb{Q}_0^+, +)$ generated by A, whose definition is the same of that used in the previous section. Clearly, $S(A) = \langle A \rangle \cap \mathbb{N}$ is a submonoid of $\mathbb{N}$. It is proved in [52] that if a, b and c are positive integers with $c < a < b$, then $S(a,b,c) = S\left(\left[\frac{b}{a}, \frac{b}{a-c}\right]\right)$. Since $S(a,b,c) = \mathbb{N}$ when $a \ge c$, and the inequality $ax \bmod b \le cx$ has the same integer solutions as $(a \bmod b)x \bmod b \le cx$, the condition $c < a < b$ is not restrictive.

As a consequence of the results proved in [52], we have that a numerical semigroup S is proportionally modular if and only if there exist two positive rational numbers $\alpha < \beta$ such that $S = S([\alpha, \beta])$. This is also equivalent to the existence of an interval I, with nonempty interior, of the form $S = S(I)$ (see [55]).

By using the notation introduced in [54], a sequence of fractions $a_1/b_1 < a_2/b_2 < \cdots < a_p/b_p$ is said to be a Bézout sequence if $a_1, \dots, a_p$, $b_1, \dots, b_p$ are positive integers and $a_{i+1}b_i - a_i b_{i+1} = 1$ for all $i \in \{1, \dots, p-1\}$. The importance of the Bézout sequences in the study of proportionally modular semigroups highlights in the following result proved in [54]. If $a_1/b_1 < a_2/b_2 < \cdots < a_p/b_p$ is a Bézout sequence, then $S([a_1/b_1, a_p/b_p]) = \langle a_1, \dots, a_p \rangle$.

A Bézout sequence $a_1/b_1 < a_2/b_2 < \cdots < a_p/b_p$ is proper if $a_{i+h}b_i - a_i b_{i+h} \ge 2$ for all $h \ge 2$ with $i, i+h \in \{1, \dots, p\}$. Clearly, every Bézout sequence can be reduced (by removing some terms) to a proper Bézout sequence with the same ends as the original one. It is showed in [9], that if $a_1/b_1 < a_2/b_2$ are two reduced fractions, then there exists an unique proper Bézout sequence with ends a_1/b_1 and a_2/b_2. Furthermore, in this work a procedure for obtaining this sequence is given.

It is proved in [54] that if $a_1/b_1 < a_2/b_2 < \cdots < a_p/b_p$ is a proper Bézout sequence, then there exists $h \in \{1, \dots, p\}$ such that $a_1 \ge \cdots \ge a_h \le \cdots \le a_p$ (the sequence $a_1, \dots, a_p$ is convex). The following characterization is also proved there: a numerical semigroup is proportionally modular if and only if there exists a convex ordering if its minimal generators $n_1, \dots, n_e$ such that $\gcd\{n_i, n_{i+1}\} = 1$ for all $i \in \{1, \dots, e-1\}$ and $n_{j-1} + n_{j+1} \equiv 0 \pmod{n_j}$ for all $j \in \{2, \dots, e-1\}$.

A modular Diophantine inequality is a proportionally modular Diophantine inequality with proportion equal to one. A numerical semigroup is said to be modular if it is the set of solutions of some modular Diophantine inequality. Clearly, every modular numerical semigroup is proportionally modular, and this inclusion is strict as it is proved in [52]. A formula for $g(S(a,b,1))$ in function of a and b is given in [53]. The problems of finding formulas for $F(S(a,b,1))$, $m(S(a,b,1))$, $t(S(a,b,1))$ and $e(S(a,b,1))$ remain open. It is not known if the mentioned conjecture of Wilf is true for modular semigroups neither.

A semigroup of the form $\{0, m, \rightarrow\}$ is said to be ordinary. A numerical semigroup S is an open modular numerical semigroup if it is ordinary or of it is the form $S = S(]b/a, b/(a-1)[)$ for some integers $2 \le a < b$. Therefore these semigroups are proportionally modular. Moreover, it is proved in [55] that every proportionally modular numerical semigroup can be expressed as a finite intersection of open modular numerical semigroups. The formulas for the Frobenius number, the genus and the type of open modular semigroups are also obtained in the just quoted work. However the rest of the problems previously suggested for modular numerical semigroups remain still open.

As we mentioned above, a characterization for proportionally modular numerical semigroups in terms of its systems of minimal generators is given in [54]. The question of giving formulas for the Frobenius number, genus and type of a proportionally modular numerical semigroup in terms of its system of minimal generators remains unsolved too.

Following the terminology in [57], a contracted modular Diophantine inequality is an expression of the form $ax \bmod b \le x - c$, where a, b and c are nonnegative integers and $b \ne 0$. Let us denote by $T(a,b,c)$ the set of integer solutions of the last inequality. Then $T(a,b,c) \cup \{0\}$ is a numerical semigroup. An algorithm that allows us to determine whether a semigroup is the set of solutions of a contracted modular Diophantine equation is given in [57]. A formula for the genus of $T(a,b,c) \cup \{0\}$ is also given there.

A contracted proportionally modular Diophantine inequality is an expression of the form $ax \bmod b \le cx - d$, with a, b, c and d nonnegative integers and $b \ne 0 \ne c$. If we denote by $P(a,b,c,d)$ the set of solutions of such an inequality, then $P(a,b,c,d) \cup \{0\}$ is a numerical semigroup. It is not yet known an algorithm to determine whether a semigroup is of this form.

The Stern-Brocot tree gives a recursive method for constructing all the reduced fractions x/y, with x and y positive integers (see [20]). For constructing this tree we start with the expressions $0/1$ and $1/0$. In each step of the process we insert between each two consecutive expressions m/n and m'/n' its median $(m+m')/(n+n')$. We obtain in this way the sequences

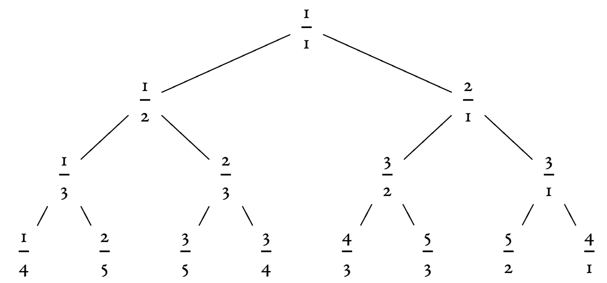
$$\frac{0}{1} < \frac{1}{1} < \frac{1}{0}$$

$$\frac{0}{1} < \frac{1}{2} < \frac{1}{1} < \frac{2}{1} < \frac{1}{0}$$

$$\frac{0}{1} < \frac{1}{3} < \frac{1}{2} < \frac{2}{3} < \frac{1}{1} < \frac{3}{2} < \frac{2}{1} < \frac{3}{1} < \frac{1}{0}$$

$$\cdots$$

The Stern-Brocot tree can now be obtained by connecting each median with the fractions used for computing it and being in the previous level but not in the levels above it.



It is proved in [9] that if m/n is the common predecessor of two fractions $a/b < c/d$ in the Stern-Brocot tree, then m is the multiplicity of $S([a/b, c/d])$. It could be nice to obtain other constants of the semigroup by looking at this tree.

3. The quotient of a numerical semigroup by a positive integer

Let S be a numerical semigroup and p be a positive integer. Let us denote by

$$\frac{S}{p} = \{x \in \mathbb{N} \mid px \in S\}.$$

Clearly, S/p is a numerical semigroup, and we will call it the quotient of S by p. According to this notation, we will call $S/2$ one half of S and that $S/4$ is a quarter of S. These two cases will have an special importance in this section.

It is proved in [56] that a numerical semigroup is proportionally modular if and only if it is the quotient of an embedding dimension two numerical semigroup by a positive integer. This result is improved in [32] by proving that a numerical semigroup is proportionally modular if and only if it is of the form $\langle a, a+1 \rangle / d$ with a and d positive integers. We still do not have formulas for $F(\langle a, a+1 \rangle / d)$, $g(\langle a, a+1 \rangle / d)$, $m(\langle a, a+1 \rangle / d)$, $t(\langle a, a+1 \rangle / d)$ and $e(\langle a, a+1 \rangle / d)$.

The next step in this line of research would be studying those numerical semigroups that are the quotient of a numerical semigroup with embedding dimension three by a positive integer. Unfortunately we do not have a procedure that allows us to distinguish such a semigroup from the rest. Moreover, we still do not know of any example of semigroups that are not of this form.

A numerical semigroup S is symmetric if $x \in \mathbb{Z} \setminus S$ implies $F(S) - x \in S$. These semigroups have been widely studied. Their main motivation comes from a work by Kunz ([26]) from which it can be deduced that a numerical semigroup is symmetric if and only if its associated numerical semigroup ring is Gorenstein. Symmetric numerical semigroups always have odd Frobenius number, thus for numerical semigroups with even Frobenius number, the equivalent notion to symmetric semigroups is that of pseudo-symmetric numerical semigroups. We say that S is a pseudo-symmetric numerical semigroup if it has even Frobenius number and for all $x \in \mathbb{Z} \setminus S$, we have either $F(S) - x \in S$ or $x = F(S)/2$. The concept of irreducible semigroup, introduced in [40], collects these two families of semigroups. A numerical semigroup is irreducible if it cannot be expressed as the intersection of two semigroups that contain it properly. It can be proved that a semigroup is irreducible if and only if it is either symmetric (with odd Frobenius number) or pseudo-symmetric (with even Frobenius number).

Intuition (and the tables of the number of numerical semigroups with a given genus or Frobenius number we have) tells us that the percentage of irreducible numerical semigroups is quite small. It is proved in [44] that every numerical semigroup is one half of an infinite number of symmetric numerical semigroups. The apparent parallelism between symmetric and pseudo-symmetric numerical semigroups fails as we can see in [37], where it is proved that a numerical semigroup is irreducible if and only it is one half of a pseudo-symmetric numerical semigroup. As a consequence we have that every numerical semigroup is a quarter of infinitely many pseudo-symmetric numerical semigroups. In [61], it is also shown that for every positive integer d and every numerical semigroup S, there exist infinitely many symmetric numerical semigroups T such that $S = T/d$, and if $d \geq 3$, then there exist infinitely many pseudo-symmetric numerical semigroups T with $S = T/d$.

From the definition, we deduce that a numerical semigroup S is symmetric if and only if $\mathrm{g}(S) = (\mathrm{F}(S) + 1)/2$. Therefore these numerical semigroups verify Wilf's conjecture previously mentioned. We raise the following question. If a numerical semigroup verifies Wilf's conjecture, then does so its half?

It can easily be seen that every numerical semigroup can be expressed as a finite intersection of irreducible numerical semigroups. A procedure for obtaining such a decomposition is given in [50]. Furthermore it is also explained how to obtain a decomposition with the least possible number of irreducibles. We still do not know how many numerical semigroups appear in these minimal decompositions, moreover, we wonder if there exists a positive integer N such that every numerical semigroup can be expressed as an intersection of at most N irreducible numerical semigroups.

In [62] Toms introduces a class of numerical semigroups that are the positive cones of the K_0 groups of certain C^*-algebras. Given a numerical semigroup we say, inspired in this work, that it admits a Toms decomposition if and only if there exist positive integers $q_1, \dots, q_n, m_1, \dots, m_n$ and L such that $\gcd\{q_i, m_i\} = \gcd\{L, m_i\} = \gcd\{L, q_i\} = 1$ for all $i \in \{1, \dots, n\}$ and $S = (1/L) \cap_{i=1}^{n} \langle q_i, m_i \rangle$.

As $(1/L) \cap_{i=1}^{n} \langle q_i, m_i \rangle = \cap_{i=1}^{n} \langle q_i, m_i \rangle / L$, we have that if a numerical semigroup admits a Toms decomposition, then S is a finite intersection of proportionally modular numerical semigroups. It is proved in [46] that the reciprocal is also true. Therefore, a numerical semigroup admits a Toms decomposition if and only if it is an intersection of finitely many proportionally modular numerical semigroups. These kind of semigroups are studied in [14], where an algorithm for distinguishing whether a numerical semigroup is an intersection of finitely many proportionally modular numerical semigroups is given. Furthermore, in the affirmative case it gives us a minimal decomposition, and in the negative case it gives us the least numerical semigroup which is intersection of proportionally modular semigroups and contains the original numerical semigroup (its proportionally modular closure).

It is conjectured in [57] that every contracted modular numerical semigroup admits a Toms decomposition.

Note that the numerical semigroups that admit a Toms decomposition are those that are the set of solutions of a system of proportionally modular Diophantine inequalities. It is proved in [32] that two systems of inequalities are always equivalent to another system with all the inequalities having the same modulus, which moreover can be chosen to be prime. Now we raise the following question: is every system of proportionally modular Diophantine inequalities equivalent to a system with all proportions being equal to one?, or equivalently, if a numerical semigroup admits a Tom decomposition, can it be expressed as an intersection of modular numerical semigroups?

Following the terminology introduced in [51], a gap x in a numerical semigroup S is said to be fundamental if $\{2x, 3x\} \subset S$ (and therefore $kx \in S$ for every integer with $k \geq 2$). Let us denote by $\mathrm{FG}(S)$ the set of all fundamental gaps of S. If $X \subseteq \mathbb{Z}$, then $\mathrm{D}(X)$ will denote the union of all positive divisors of the elements of X. It can easily be shown that $S = \mathbb{N} \setminus \mathrm{D}(\mathrm{FG}(S))$. Therefore, a way to represent a semigroup is by giving its fundamental gaps. This representation is specially useful when studying the quotient of a semigroup S by a positive integer d, since $\mathrm{FG}(S/d) = \{h/d \mid h \in \mathrm{FG}(S),\ h \equiv 0 \pmod{d}\}$.

The cardinality of the set of fundamental gaps of a semigroup is an invariant of the semigroup. We can therefore open a new line of research by studying numerical semigroups attending to their number of fundamental gaps. It would be also interesting to find simple sufficient conditions that allow us to decide when a subset X of $\mathbb{N}$ is the set of fundamental gaps of some numerical semigroup.

Let S be a numerical semigroup. In [33] the set T of all numerical semigroups such that $S = T/2$ is studied, the semigroup of the "doubles" of S. In the just quoted work we raise the question of finding a formula that depends on S and allows us to compute the minimum of the Frobenius numbers of the doubles of S.

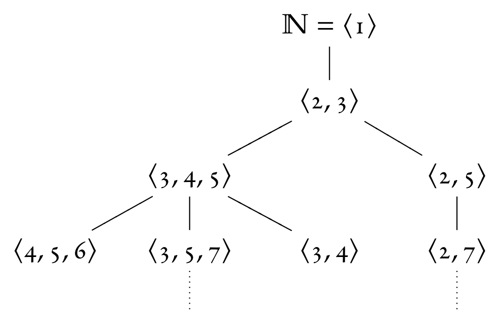
Following this line we can ask ourselves about the set of all "triples" (or multiples in general) of a numerical semigroup.

Finally, it would be interesting to characterize the families of numerical semigroups verifying that any of its elements can be realized as a quotient of some element of the family by a fixed positive integer.

4. Frobenius Varieties

A directed graph G is a pair (V, E), where V is a nonempty set whose elements are called vertices, and E is a subset of $\{(u, v) \in V \times V \mid u \neq v\}$. The elements of E are called edges of the graph. A path connecting two vertices x and y of G is a sequence of distinct edges of the form $(v_0, v_1), (v_1, v_2), \dots, (v_{n-1}, v_n)$ with $v_0 = x$ and $v_n = y$. A graph G is a tree if there exists a vertex r (called the root of G) such that for any other vertex x of G, there exists an unique path connecting x and r. If (x, y) is an edge of the tree, then x is a son of y. A vertex of a tree is a leaf if it has no sons.

Let $\mathscr{S}$ be the set of all numerical semigroups. We define the graph associated to $\mathscr{S}$, $\mathscr{G}(\mathscr{S})$, to be the graph whose vertices are all the elements of $\mathscr{S}$ and $(T, S) \in \mathscr{S} \times \mathscr{S}$ is an edge if $S = T \cup \{\mathrm{F}(T)\}$. In [45], it is proved that $\mathscr{G}(\mathscr{S})$ is a tree with root $\mathbb{N}$, and that the sons of $S \in \mathscr{S}$ are the subsets $S \setminus \{x_1\}, \dots, S \setminus \{x_r\}$, where $x_1, \dots, x_r$ are the minimal generators of S greater than $\mathrm{F}(S)$. Therefore S is a leaf of $\mathscr{G}(\mathscr{S})$ if it has no minimal generators greater than $\mathrm{F}(S)$. These results allow us to construct recursively the set of numerical semigroups starting with $\mathbb{N}$.



The level of a vertex in a directed graph is the length of the path connecting this vertex with the root. Note that in $\mathscr{G}(\mathscr{S})$ the level of a vertex coincides with its genus as numerical semigroup. Therefore, the Bras-Amorós' conjecture quoted in the end of the first section can be reformulated by saying that in $\mathscr{G}(\mathscr{S})$ there are more vertices in the $(n+1)$th level than in the nth one.

A Frobenius variety is a nonempty family $\mathscr{V}$ of numerical semigroups such that

1) if $S, T \in \mathscr{V}$, then $S \cap T \in \mathscr{V}$,
2) if $S \in \mathscr{V}$, $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)\} \in \mathscr{V}$.

The concept of Frobenius variety was introduced in [38] with the aim of generalizing most of the results in [6, 14, 48, 49]. In particular, the semigroups that belong to a Frobenius variety can be arranged as a directed tree with similar properties to those of $\mathscr{G}(\mathscr{S})$.

Clearly, $\mathscr{S}$ is a Frobenius variety. If $A \subseteq \mathbb{N}$, then $\{S \in \mathscr{S} \mid A \subseteq S\}$ is also a Frobenius variety. In particular, $\mathscr{O}(S)$, the set of all numerical semigroups that contain S, is a Frobenius variety. We next give some interesting examples of Frobenius varieties.

Inspired by [1], Lipman introduces and motivates in [27] the study of Arf rings. The characterization of them via their numerical semigroup of values, brings us to the following concept: a numerical semigroups S is said to be Arf if for every $x, y, z \in S$, with $x, y \geq z$ we have $x + y - z \in S$. It is proved in [48] that the set of Arf numerical semigroups is a Frobenius variety.

Saturated rings were introduced independently in three distinct ways by Zariski ([67]), Pham-Teissier ([29]) and Campillo ([10]), although the definitions given in these works are equivalent on algebraically closed fields of characteristic zero. Like in the case of numerical semigroups with the Arf property, saturated numerical semigroups appear when characterizing these rings in terms of their numerical semigroups of values. A numerical semigroup S is saturated if for every $s, s_1, \dots, s_r \in S$ with $s_i \leq s$ for all $i \in \{1, \dots, r\}$ and $z_1, \dots, z_r \in \mathbb{Z}$ being integers such that $z_1 s_1 + \cdots + z_r s_r \geq 0$, then we have $s + z_1 s_1 + \cdots + z_r s_r \in S$. It is proved in [49] that the set of saturated numerical semigroups is a Frobenius variety.

The class of Arf and Saturated numerical semigroups is also closed under quotients by positive integers as shown in [17], though the larger class of maximal embedding dimension numerical semigroups is not (if S is a numerical semigroup, then $\mathrm{e}(S) \leq \mathrm{m}(S)$; a numerical semigroup is said to be a maximal embedding dimension semigroup, or to have maximal embedding dimension, if $\mathrm{e}(S) = \mathrm{m}(S)$). What is the Frobenius variety generated by maximal embedding dimension numerical semigroups?

As a consequence of [14] and [46], it can be deduced that the set of numerical semigroups that admit a Toms decomposition is a Frobenius variety. Every semigroup with embedding dimension two admits a Toms decomposition. Is the variety of numerical semigroups admitting a Toms decomposition the least Frobenius variety containing all semigroups with embedding dimension two?

The idea of pattern of a numerical semigroup was introduced in [6] with the aim of trying to generalize the concept of Arf numerical semigroup. A pattern P of length n is a linear homogeneous polynomial with non-zero integer coefficients in $x_1, \dots, x_n$ (for $n = 0$ the only pattern is $p = 0$). We will say that numerical semigroup S admits a pattern $a_1 x_1 + \dots + a_n x_n$ if for every sequence $s_1 \geq s_2 \geq \cdots \geq s_n$ of elements in S, we have

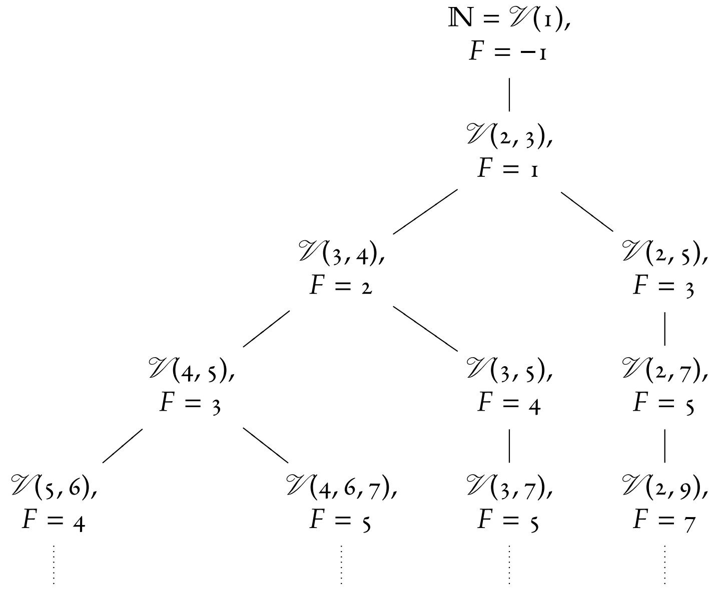


Figure 1

$a_1s_1 + \cdots + a_ns_n \in S$. We denote by $\mathscr{S}_P$ the set of all numerical semigroups that admit a pattern P. Then the set of numerical semigroups with the Arf property is $\mathscr{S}_{x_1+x_2-x_3}$. It is proved in [6] that for every pattern P of a special type (strongly admissible), $\mathscr{S}_P$ is a Frobenius variety. What varieties arise in this way? It would be interesting to give a weaker definition of pattern such that every variety becomes the variety associated to a pattern.

The intersection of Frobenius varieties is again a Frobenius variety. This fact allows us to construct new Frobenius varieties from known Frobenius varieties and moreover, it allows us to talk of the Frobenius variety generated by a family X of numerical semigroups. This variety will be denoted by $\mathscr{F}(X)$, and it is defined to be the intersection of all Frobenius varieties containing X. If X is finite, then $\mathscr{F}(X)$ is finite and it is shown in [38] how to compute all the elements of $\mathscr{F}(X)$.

Let $\mathscr{V}$ be a Frobenius variety. A submonoid M of $\mathbb{N}$ is a $\mathscr{V}$-monoid if it can be expressed as an intersection of elements of $\mathscr{V}$. It is clear that the intersection of $\mathscr{V}$-monoids is again a $\mathscr{V}$-monoid. Thus given $A \subseteq \mathbb{N}$ we can define the $\mathscr{V}$-monoid generated by A as the intersection of all $\mathscr{V}$-monoids containing A. We will denote by $\mathscr{V}(A)$ this $\mathscr{V}$-monoid and we will say that A is a $\mathscr{V}$-system of generators of it. If there is no proper subset of A being a $\mathscr{V}$-system of generators $\mathscr{V}(A)$, then A is a minimal $\mathscr{V}$-system of generators of $\mathscr{V}(A)$. It is proved in [38] that every $\mathscr{V}$-monoid admits an unique minimal $\mathscr{V}$-system of generators, and that moreover this system is finite.

We define the directed graph $\mathscr{G}(\mathscr{V})$ in the same way we defined $\mathscr{G}(\mathscr{S})$, that is, as the graph whose vertices are the elements of $\mathscr{V}$, and $(T, S) \in \mathscr{V} \times \mathscr{V}$ is an edge of the above graph if $S = T \cup \{\mathrm{F}(T)\}$. This graph is a tree with root $\mathbb{N}$ ([38]). Moreover, the sons of a semigroup S in $\mathscr{V}$ are $S \setminus \{x_1\}, \dots, S \setminus \{x_r\}$, where $x_1, \dots, x_r$ are the minimal $\mathscr{V}$-generators of S greater than $\mathrm{F}(S)$. This fact allows us to find all the elements of the variety $\mathscr{V}$ from $\mathbb{N}$.

Figure 1 represents part of the tree associated to the variety of numerical semigroups with the Arf property.

Figure 2 represents part of the tree corresponding to saturated numerical semigroups.

As a generalization of Bras-Amorós' conjecture, we can raise the following question. If $\mathscr{V}$ is a Frobenius variety, does there exist on $\mathscr{G}(\mathscr{V})$ more vertices in the $(n+1)$th level than in the nth one? The answer to this question is no, as it is proved in [38, Example 26]. However, the same question in the case of $\mathscr{V}$ being infinite remains open. Another interesting question would be characterizing those Frobenius varieties that verify the Bras-Amorós' conjecture.

If $\mathscr{V}$ is a Frobenius variety and $S \in \mathscr{V}$, then it is known that S admits an unique minimal $\mathscr{V}$-system of generators, and moreover it is finite. The cardinality of

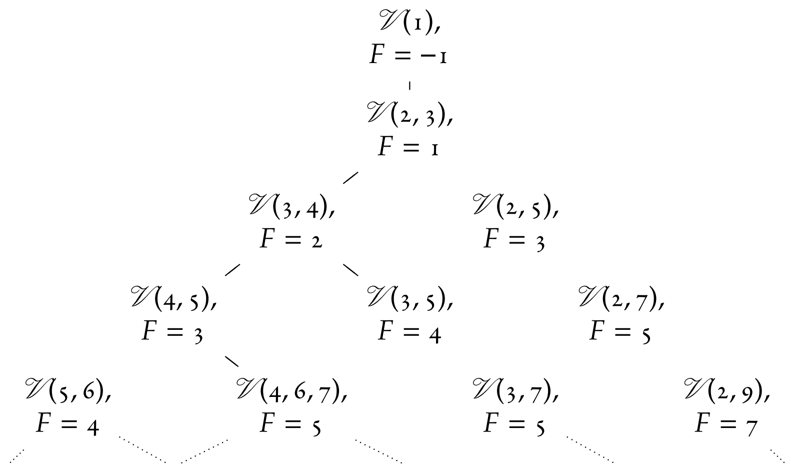


Figure 2

the set above is an invariant of S that will be called the embedding $\mathscr{V}$-dimension of S, and it will be denoted by $\mathrm{e}_{\mathscr{V}}(S)$. As a generalization of Wilf's conjecture, we would like to characterize those Frobenius varieties $\mathscr{V}$ such that for every $S \in \mathscr{V}$, then $\mathrm{e}_{\mathscr{V}}(S)\mathrm{g}(S) \leq (\mathrm{e}_{\mathscr{V}}(S) - 1)(\mathrm{F}(S) + 1)$.

Clearly, the Frobenius variety generated by irreducible numerical semigroups is $\mathscr{S}$, the set of all numerical semigroups. What is the Frobenius variety generated only by the symmetric ones? and by the pseudo-symmetric ones?

5. Presentations of a numerical semigroup

Let $(S, +)$ be a commutative monoid. A congruence σ over S is an equivalence relation that is compatible with addition, that is, if $a\sigma b$ with $a, b \in S$, then $(a+c)\sigma(b+c)$ for all $c \in S$. The set S/σ endowed with the operation $[a] + [b] = [a+b]$ is a monoid. We will call it the quotient monoid of S by σ.

If S is generated by $\{s_1, \dots, s_n\}$, then the map $\varphi : \mathbb{N}^n \to S$, $(a_1, \dots, a_n) \mapsto a_1s_1 + \cdots + a_ns_n$ is a monoid epimorphism. Therefore S is isomorphic to $\mathbb{N}^n/\sim_S$, where $\sim_S$ is the kernel congruence of φ, that is, $a \sim_S b$ if $\varphi(a) = \varphi(b)$.

The intersection of congruences over a monoid S is again a congruence over S. This fact allows us, given $\sigma \subseteq S \times S$, to define the concept of congruence generated by σ as the intersection of all congruences over S containing σ, and it will be denoted by $\langle\sigma\rangle$.

Rédei proves in [31] that every congruence over $\mathbb{N}^n$ is finitely generated, that is, there exists a subset of $\mathbb{N}^n \times \mathbb{N}^n$ with finitely many elements generating it. As a consequence we have that giving a finitely generated monoid is, up to isomorphism, equivalent to giving a finite subset of $\mathbb{N}^n \times \mathbb{N}^n$.

If S is a numerical semigroup with minimal generators system $\{n_1, \dots, n_e\}$, then there exists a finite subset σ of $\mathbb{N}^e \times \mathbb{N}^e$ such that S is isomorphic to $\mathbb{N}^e/\langle\sigma\rangle$. We say that σ is a presentation of S. If moreover σ has the least possible cardinality, then σ is a minimal presentation of S.

A (non directed) graph G is a pair (V, E), where V is a nonempty set of elements called vertices, and E is a subset of $\{\{u, v\} \mid u, v \in V, u \neq v\}$. The non ordered pair $\{u, v\}$ will be denoted by $\overline{uv}$, and if it belongs to E, then we say that it is an edge of G. A sequence of the form $\overline{v_0v_1}, \overline{v_1v_2}, \dots, \overline{v_{m-1}v_m}$ is a path of length m connecting the vertices v_0 and v_m. A graph is connected if any two distinct vertices are connected by a path. A graph $G' = (V', E')$ is said to be a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$. A connected component of G is a maximal connected subgraph of G. It is well known (see for instance [28]) that a connected graph with n vertices has at least $n-1$ edges. A (finite) tree with n vertices is a connected graph with $n-1$ edges.

Let us remind now the method described in [35] for computing the minimal presentation of a numerical semigroup. Let S be a numerical semigroup with minimal system of generators $\{n_1, \dots, n_e\}$. For each $n \in S$, let us define $G_n = (V_n, E_n)$, where $V_n = \{n_i \mid n - n_i \in S\}$ and $E_n = \{\overline{n_i n_j} \mid n - (n_i + n_j) \in S, i \neq j\}$. If G_n is connected, we take $\sigma_n = \emptyset$. If G_n is not connected and $V_1, \dots, V_r$ are the sets of vertices corresponding to the connected components in G_n, then we define $\sigma_n = \{(\alpha_1, \alpha_2), (\alpha_1, \alpha_3), \dots, (\alpha_1, \alpha_r)\}$, where $\alpha_i \in \varphi^{-1}(n)$ and its j-th component is zero whenever $n_j \notin V_i$. It is proved in [35] that $\sigma = \cup_{n \in S} \sigma_n$ is a minimal presentation for S. Let us notice that the set $\mathrm{Betti}(S) = \{n \in S \mid G_n \text{ is not connected}\}$ is finite, and that its cardinality is an invariant of S. A line of research could be the study of Betti(S), and its relation with other invariants of S mentioned above. In [19] affine semigroups (and thus numerical semigroups) with a single Betti element are studied. What are those numerical semigroups having two or three Betti elements?

It is also shown in [35] how all the minimal presentations of a semigroup are. In particular, we can determine whether a numerical semigroup admits a unique minimal presentation. Motivated by the idea of generic ideal, we may ask what are the numerical semigroups that admit a unique minimal presentation, and characterize them in terms of their minimal generators.

If S is a numerical semigroup, then the cardinality of a minimal presentation of S is greater than or equal to $\mathrm{e}(S) - 1$. Those semigroups that attain this bound are said to be complete intersections. This kind of semigroup has been well studied, and Delorme gives in [15] a good characterization of them. Every numerical semigroup with embedding dimension two is a complete intersection, and every complete intersection is symmetric (see [21]). We raise the following questions. What semigroups can be expressed as the quotient of a complete intersection by a positive integer? What is the least Frobenius variety containing all the complete intersection numerical semigroups?

Let S_1 and S_2 be two numerical semigroups minimally generated by $\{n_1, \dots, n_r\}$ and $\{n_{r+1}, \dots, n_e\}$, respectively. Let $\lambda \in S_1 \setminus \{n_1, \dots, n_r\}$ and $\mu \in S_2 \setminus \{n_{r+1}, \dots, n_e\}$, such that $\gcd\{\lambda, \mu\} = 1$. We then say that $S = \langle \mu n_1, \dots, \mu n_r, \lambda n_{r+1}, \dots, \lambda n_e \rangle$ is a gluing to S_1 and S_2. It is proved in [45] how given minimal presentations of S_1 and S_2, one easily gets a minimal presentation of S. The characterization given by Delorme in [15], with this notation, can be reformulated in the following way: a numerical semigroup is a complete intersection if and only if is a gluing to two numerical semigroups that are a complete intersection. A consequence of this result is that the set of semigroups that are a complete intersection is the least family of numerical semigroups containing $\mathbb{N}$ being closed under gluing. It is well known that the family of numerical symmetric semigroups is also closed under gluing ([45]). It would be interesting to study other families closed under gluing. Which is the least family containing those semigroups with maximal embedding dimension and closed under gluing?

Bresinsky gives in [7] a family of numerical semigroups with embedding dimension four and with cardinality of its minimal presentations arbitrarily large. This fact proves that the cardinality of a minimal presentation of a numerical semigroup cannot be upper bounded just in function of its embedding dimension. Bresinski also proves in [8] that the cardinality for a minimal presentation of a symmetric numerical semigroup with embedding dimension four can only be three or five. It is conjectured in [36] that if S is a symmetric numerical semigroup with $\mathrm{e}(S) \geq 3$, then the cardinality of a minimal presentation for S is less than or equal to $\mathrm{e}(S)(\mathrm{e}(S) - 1)/2 - 1$. Barucci [2] proves with the semigroup $\langle 19, 23, 29, 31, 37 \rangle$ that the conjecture above is not true. However, the problem of determining if the cardinality of a minimal presentation of a symmetric numerical semigroup can be bounded in function of the embedding dimension remains open.

Let σ be a finite subset of $\mathbb{N}^n \times \mathbb{N}^n$. By using the results in [41, 45] it is possible to determine algorithmically whether $\mathbb{N}^n/\langle\sigma\rangle$ is isomorphic to a numerical semigroup. However we miss in the literature families of subsets σ of $\mathbb{N}^n$ so that we can assert, without using algorithms, that $\mathbb{N}^n/\langle\sigma\rangle$ is isomorphic to a numerical semigroup. More specifically, we suggest the following problem: given

$$\sigma = \{((c_1, 0, \dots, 0), (0, a_{1_1}, \dots, a_{1_n})), \dots, ((0, \dots, c_n), (a_{n_1}, \dots, a_{n_{n-1}}, 0))\},$$

which conditions the integers c_i and a_{j_k} have to verify so that $\mathbb{N}^n/\langle\sigma\rangle$ is isomorphic to a numerical semigroup? Herzog proved in [21] that embedding dimension three numerical semigroups always have a minimal presentation of this form. Neat numerical semigroups introduced by Komeda in [24] are also of this form.

6. Numerical semigroups with embedding dimension three

Herzog proves in [21] that a numerical semigroup with embedding dimension three is symmetric if and only if it is a complete intersection. This fact allows us to characterize symmetric numerical semigroups with embedding dimension three in the following way (see [45]). A numerical semigroup S with $\mathbf{e}(S) = 3$ is symmetric if and only if $S = \langle am_1, am_2, bm_1 + cm_2 \rangle$, with a, b, c, m_1 and m_2 nonnegative integers, such that m_1, m_2, a and $b + c$ are greater than or equal to two and $\gcd\{m_1, m_2\} = \gcd\{a, bm_1 + cm_2\} = 1$. Moreover, as it is proved in [45],

$$\begin{aligned}\mathrm{F}(\langle am_1, am_2, bm_1 + cm_2 \rangle) &= \\ &= a(m_1 m_2 - m_1 - m_2) + (a-1)(bm_1 + cm_2).\end{aligned}$$

We also have a formula for the genus, since S is symmetric, $\mathrm{g}(S) = (\mathrm{F}(S) + 1)/2$. Finally, we also know the type, since it is proved in [18] that a numerical semigroup is symmetric if and only if its type is equal to one.

We study in [43] the set of pseudo-symmetric numerical semigroups with embedding dimension three. In particular, we give the following characterization. A numerical semigroup S with $\mathrm{e}(S) = 3$ is pseudo-symmetric if and only if for some ordering of its minimal generators, by taking

$$\Delta = \sqrt{(\textstyle\sum n_i)^2 - 4(n_1 n_2 + n_1 n_3 + n_2 n_3 - n_1 n_2 n_3)},$$

then

$$\left\{\frac{n_1 - n_2 + n_3 + \Delta}{2n_1}, \frac{n_1 + n_2 - n_3 + \Delta}{2n_2}, \frac{-n_1 + n_2 + n_3 + \Delta}{2n_3}\right\} \subset \mathbb{N}.$$

Moreover, in this case, $\mathrm{F}(\langle n_1, n_2, n_3 \rangle) = \Delta - (n_1 + n_2 + n_3)$. We also know the genus and the type, since if S is a pseudo-symmetric numerical semigroups, then $\mathrm{g}(S) = (\mathrm{F}(S) + 2)/2$ and by [18], $\mathrm{t}(S) = 2$.

Bresinsky ([7]) and Komeda ([24]) fully characterize those symmetric and pseudo-symmetric numerical semigroups, respectively, with embedding dimension four. They show that their minimal presentations always have cardinality five.

Curtis proves in [13] the impossibility of giving an algebraic formula for the Frobenius number of a numerical semigroup in terms of its minimal generators on embedding dimension three. We raise the following question. Given a polynomial $f(x_1, x_2, x_3, x_4) \in \mathbb{Q}[x_1, x_2, x_3, x_4]$, study the family of numerical semigroups S such that if S is minimally generated by $n_1 < n_2 < n_3$, and F is the Frobenius number of S, then $f(n_1, n_2, n_3, F) = 0$.

Our aim now is studying the set of numerical semigroups with embedding dimension three in general. By [18], we know that these semigroups have type one or two, and by using [22, 34] if we are concerned with the Frobenius number and the genus, we can focus ourselves in those numerical semigroups whose minimal generators are pairwise relatively prime. The following result appears in [42]. Let n_1, n_2 and n_3 three pairwise relatively prime positive integers. Then the system of equations

$$\begin{aligned} n_1 &= r_{12} r_{13} + r_{12} r_{23} + r_{13} r_{32}, \\ n_2 &= r_{13} r_{21} + r_{21} r_{23} + r_{23} r_{31}, \\ n_3 &= r_{12} r_{31} + r_{21} r_{32} + r_{31} r_{32}. \end{aligned}$$

has a (unique) positive integer solution if and only if $\{n_1, n_2, n_3\}$ generates minimality $\langle n_1, n_2, n_3 \rangle$. In [42] the authors give formulas for the pseudo-Frobenius number and the genus of $\langle n_1, n_2, n_3 \rangle$ from the solutions of the above system. Thus it seems natural to ask, given positive integers r_{ij}, with $i, j \in \{1, 2, 3\}$, when $r_{12} r_{13} + r_{12} r_{23} + r_{13} r_{32}$, $r_{13} r_{21} + r_{21} r_{23} + r_{23} r_{31}$ and $r_{12} r_{31} + r_{21} r_{32} + r_{31} r_{32}$ are pairwise relatively prime?

Let S be a numerical semigroup minimally generated by three positive integers n_1, n_2 and n_3 being pairwise relatively prime. For each $i \in \{1, 2, 3\}$, let $c_i = \min\{x \in \mathbb{N} \setminus \{0\} \mid x n_i \in \langle \{n_1, n_2, n_3\} \setminus \{n_i\} \rangle\}$. In [42] formulas for $\mathrm{F}(S)$ and $\mathrm{g}(S)$ from n_i and c_i ($i \in \{1, 2, 3\}$) are given. Therefore, if we had a formula for computing c_3 from n_1 and n_2, we would have solved the problems raised by Frobenius for embedding dimension three. Note that c_3 is nothing but the multiplicity of the proportionally modular semigroup $\langle n_1, n_2 \rangle / n_3$. It is proved in [58] that if u is a positive integer such that $u n_2 \equiv 1 \pmod{n_1}$, then $\langle n_1, n_2 \rangle / n_3 = \{x \in \mathbb{Z} \mid u n_2 n_3 x \bmod n_1 n_2 \leq n_3 x\}$. We suggest in this line the problem of finding a formula that allows us to give the multiplicity of $\mathrm{S}(u n_2 n_3, n_1 n_2, n_3)$ from n_1, n_2 and n_3.

Fermat's Last Theorem asserts that for any integer $n \geq 3$, the Diophantine equation $x^n + y^n = z^n$ does not admit an integer solution such that $xyz \neq 0$. As it is well known, this theorem was proved by Wiles, with the help of Taylor, in 1995 ([64, 65]) after 300 years of fruitless attempts. Let us observe that for $n \geq 3$, the Diophantine equation $x^n + y^n = z^n$ has no solution verifying $zyz \neq 0$ with some of the factors equal to 1. Therefore in order to solve this equation it can be supposed that x, y and z are integers greater than or equal to two, and pairwise relatively prime. It is proved in [63], that Fermat's Last Theorem is equivalent to the following statement: if a, b and c are integers greater than or equal to two, pairwise relatively prime, and n is an integer greater than or equal to three, then the proportionally modular numerical semigroup $\langle a^n, b^n \rangle / c$ is not minimally generated by $\{a^n, c^{n-1}, b^n\}$. It would be interesting to prove this fact without using Fermat's last Theorem.

7. Non-unique factorization invariants

Let S be a numerical semigroup minimally generated by $\{n_1 < \cdots < n_e\}$. Then we already know that S is isomorphic to $\mathbb{N}^e / \sim_S$, where $\sim_S$ is the kernel congruence of the epimorphism $\varphi : \mathbb{N}^e \to S$, $(a_1, \dots, a_e) \mapsto a_1 n_1 + \cdots + a_e n_e$.

For $s \in S$, the elements in $\mathsf{Z}(s) = \varphi^{-1}(s)$ are known as factorizations of s. Given $(x_1, \dots, x_e) \in \mathbb{Z}(s)$, its length is $|x| = x_1 + \cdots + x_e$. The set of lengths of s is $\mathsf{L}(s) = \{|x| \mid x \in \mathsf{Z}(s)\}$. If $\mathsf{L}(s) = \{l_1 < l_2 < \cdots < l_t\}$, then the set of differences of lengths of factorizations of s is $\Delta(s) = \{l_2 - l_1, \dots, l_t - l_{t-1}\}$. Moreover $\Delta(S) = \bigcup_{s\in S} \Delta(s)$. These sets are known to be eventually periodic ([12]).

The elasticity of $s \in S$ is $\rho(s) = \max \mathsf{L}(s)/\min \mathsf{L}(s)$, and $\rho(S) = \sup_{s\in S}(\rho(s))$, which turns out to be a maximum ([47]). For numerical semigroups it is well known that $\rho(S) = n_e/n_1$.

For $x = (x_1, \dots, x_e), y = (y_1, \dots, y_e) \in \mathbb{N}^e$, the greatest common divisor of x and y is

$$\gcd(x,y) = (\min(x_1,y_1), \dots, \min(x_e,y_e)).$$

The distance between x and y is

$$\mathsf{d}(x,y) = \max\{|x - \gcd(x,y)|, |y - \gcd(x,y)|\}.$$

An N-chain (with N a positive integer) joining two factorizations x and y of $s \in S$ is a sequence $z_1, \dots, z_t$ of factorizations of s such that $z_1 = x$, $z_t = y$ and $\mathsf{d}(z_i, z_{i+1}) \leq N$. The catenary degree of s, $\mathsf{c}(s)$, is the least N such that for every two factorizations x and y of s, there is an N-chain joining them. The catenary degree of S is $\mathsf{c}(S) = \sup_{s\in S}\{\mathsf{c}(s)\}$. This supremum is a maximum and actually $\mathsf{c}(S) = \max_{s\in \mathrm{Betti}(S)} \mathsf{c}(s)$ ([11]). It was asked by F. Halter-Koch whether this invariant is periodic, that is, if there exists $n \in S$ such that for s "big enough", $\mathsf{c}(s+n) = \mathsf{c}(s)$.

The tame degree of $s \in S$, $\mathsf{t}(s)$, is the minimum N such that for any $i \in \{1, \dots, e\}$ with $s - n_i \in S$ and any $x \in \mathsf{Z}(s)$, there exists $y = (y_1, \dots, y_e)$, such that $y_i \neq 0$ and $\mathsf{d}(x,y) \leq N$. The tame degree of S is $\mathsf{t}(S) = \sup_{s\in S}(\mathsf{t}(S))$. This supremum is again a maximum and it is reached in the (finite) set of elements of the form $n_i + w$ with $w \in S$ such that $w - n_j \notin S$ for some $j \neq i$. F. Halter-Koch also proposed the problem of studying the eventual periodicity of S.

The invariant $\omega(S,s)$ is the least positive integer such that whenever s divides $s_1 + \cdots + s_k$ for some $s_1, \dots, s_k \in S$, then s divides $s_{i_1} + \cdots + s_{i_{\omega(S,s)}}$ for some $\{i_1, \dots, i_{\omega(S,s)}\} \subseteq \{1, \dots, k\}$. The ω-primality of S is defined as $\omega(S) = \max\{\omega(S,n_1), \dots, \omega(S,n_e)\}$. In [4] it is highlighted that numerical semigroups fulfilling $\omega(S) \neq \mathsf{t}(S)$ are rare. A characterization for numerical semigroups fulfilling this condition should be welcomed.

Another problem proposed by A. Geroldinger is to determine when can we find a numerical semigroup and an element in it with a given set of lengths.

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Tracing orbits on conservative maps

by Mário Bessa*

Abstract.—We explore uniform hyperbolicity and its relation with the pseudo orbit tracing property. This property indicates that a sequence of points which is nearly an orbit (affected with a certain error) may be shadowed by a true orbit of the system. We obtain that, when a conservative map has the shadowing property and, moreover, all the conservative maps in a C^1-small neighborhood display the same property, then the map is globally hyperbolic.

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Keywords.—Volume-preserving maps; pseudo-orbits; shadowing; hyperbolicity.

1. Introduction

> "There is strong shadow where there is much light"
>
> Goethe in Götz von Berlichingen

1.1 The basic framework

In order to start playing with dynamical systems we need a place to play and a given rule acting on it. Once we establish that, we wonder what happens when we repeat the rule ad infinitum. We are mainly interested in two types of playgrounds: volume manifolds and symplectic manifolds. On volume-manifolds the rule is the action of a volume-preserving diffeomomorphism, and on symplectic manifolds the rule is the action of a symplectomorphism. Let us now formalize these concepts.

Let M stands for a closed, connected and C^∞ Riemannian manifold of dimension $d \geq 2$ and let ν be a volume-form on M. Once we equip M with ν we denominate it by a volume-manifold. By a classic result by Moser (see [20]) we know that, in brief terms, there is only one volume-form on M. Actually, in [20] we find and atlas formed by a finite collection of smooth charts $\{\alpha_j \colon U_j \subset M \to \mathbb{R}^d\}_{j=1}^k$ where U_j are open sets and each α_j pullbacks the volume on $\mathbb{R}^d$ into ν. The volume-form allows us to define a measure μ on M which we call Lebesgue measure. A C^r $(r \geq 1)$ diffeomorphism $f : M \to M$ is said to be volume-preserving if it keeps invariant the volume structure, say $f^*\nu = \nu$. In other words any Borelian $B \subset M$ is such that $\mu(B) = \mu(f^{-1}(B))$ We denote these maps by $\mathrm{Diff}^r_\mu(M)$. We endow $\mathrm{Diff}^r_\mu(M)$ with the Whitney (or strong) C^r topology (see [1]). In broad terms, two diffeomorphisms f and g are C^r-close if they are uniformly close as well as their first C^r derivatives computed in any point $x \in M$. Such systems emerges quite naturally when considering the time-one map of incompressible flows which are a fundamental object in fluid mechanics (see e.g. [14]).

Denote by $\mathbf{M}$ a $2d$-dimensional $(d \geq 1)$ manifold with a Riemaniann structure and endowed with a closed and nondegenerate 2-form ω called symplectic form. Let μ stands for the volume measure associated to the volume form wedging ω d-times, i.e., $\nu = \omega^d = \omega \wedge \ldots \wedge \omega$. By the

* Universidade da Beira Interior, Covilhã, Portugal
bessa@ubi.pt

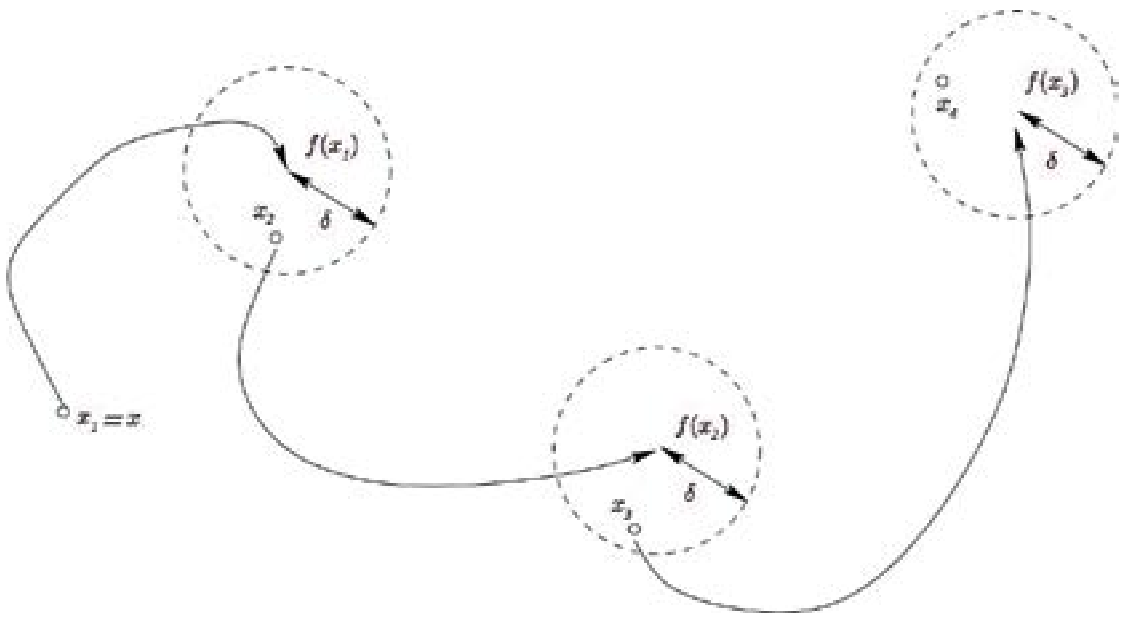

Figure 1. Illustration of a δ-pseudo-orbit

theorem of Darboux (see e.g. [21, Theorem 1.18]) there exists an atlas $\{\varphi_j\colon U_j \to \mathbb{R}^{2d}\}$, where U_j is an open subset of **M**, satisfying $\varphi_j^*\omega_0 = \omega$ with $\omega_0 = \sum_{i=1}^{d} dy_i \wedge dy_{d+i}$ being the canonical symplectic form. A diffeomorphism $f\colon \mathbf{M} \to \mathbf{M}$ is called a symplectomorphism if it leaves invariant the symplectic structure, say $f^*\omega = \omega$. Observe that, since $f^*\omega^d = \omega^d$, a symplectomorphism $f\colon \mathbf{M} \to \mathbf{M}$ preserves the volume measure μ. Moreover, in surfaces, area-preserving diffeomorphisms are symplectomorphisms since the volume-form equals the symplectic form. Symplectomorphisms arise in the classical and rational mechanics formalism as the first return Poincaré maps of hamiltonian flows. For this reason, it has long been one of the most interesting research fields in mathematical physics. We suggest the reference [21] for more details on general hamiltonian and symplectic theories. Let $\mathrm{Symp}^r_\omega(\mathbf{M})$ denote the set of all symplectomorphisms of class C^r defined on **M**. We also endow $\mathrm{Symp}^r_\omega(\mathbf{M})$ with the C^r Whitney topology.

The Riemannian structure induces a norm $\|\cdot\|$ on the tangent bundle TM and also on $T\mathbf{M}$. Denote the Riemannian distance by $d(\cdot,\cdot)$. We will use the canonical norm of a bounded linear map A given by $\|A\| = \sup_{\|v\|=1} \|A\cdot v\|$.

Given a diffeomorphism f, we denote $f^n(x) = f\circ f\circ f\circ\cdots\circ f(x)$ by composing f n-times. We say that a point p on a manifold is periodic of period $n\in\mathbb{N}$ for the diffeomorphism f if $f^n(p) = p$ and n is the minimum positive integer such that previous equality holds.

1.2 Tracing orbits and the shadowing property

The notion of shadowing in dynamical systems is inspired by the numerical computational idea of estimating differences between exact and approximate solutions along orbits and to understand the influence of the errors that we commit and allow on each iterate. We may ask if it is possible to obtain shadowing of approximate trajectories in a given dynamical system by exact ones. Nevertheless, the computational estimates, fitted with a certain error of orbits, are meaningless if they are not able to be realized by true orbits of the original system, and thus, are mere pixel imprecisions which are characteristic of the computational setup. We refer Pilyugin's book [23] for a completed description on shadowing on dynamical systems.

There are, of course, considerable limitations to the amount of information we can extract from a given specific system that exhibits the shadowing property, since a C^1-close system may be absent of that property. For this reason it is of great utility and natural to consider that a selected model can be slightly perturbed in order to obtain the same property—the stably shadowable dynamical systems.

For $\delta > 0$ and $a, b \in \mathbb{R}$ such that $a < b$, the sequence of points $\{x_i\}_{i=a}^{b}$ in M is called a δ-pseudo orbit for f if $d(f(x_i), x_{i+1}) < \delta$ for all $a \le i \le b-1$ (see Figure 1).

The diffeomorphism f is said to have the shadowing property if for all $\epsilon > 0$, there exists $\delta > 0$, such that for any δ-pseudo orbit $\{x_n\}_{n\in\mathbb{Z}}$, there is a point x which ϵ-shadows $\{x_n\}_{n\in\mathbb{Z}}$, i.e. $d(f^i(x), x_i) < \epsilon$.

Let $f \in \mathrm{Diff}^1_\mu(M)$ (respectively, $f \in \mathrm{Symp}^1_\omega(\mathbf{M})$) we say that f is C^1-stably (or robustly) shadowable if there exists a neighborhood $\mathscr{V}$ of f in $\mathrm{Diff}^1_\mu(M)$ (respectively $f \in \mathrm{Symp}^1_\omega(\mathbf{M})$) such that any $g \in \mathscr{V}$ has the shadowing property.

We point out that f has the shadowing property if and only if f^n has the shadowing property for every $n \in \mathbb{Z}$ (see [23]).

1.3 Hyperbolicity and statement of the results

Let us recall that a periodic point p of period π is said to be hyperbolic if the tangent map $Df^\pi(p)$ has no norm one eigenvalues. Being hyperbolic is stable under small C^r perturbations. The notion of hyperbolicity can be generalized to sets rather than periodic orbits.

We say that any element f in the set $\mathrm{Diff}^1_\mu(M)$ is Anosov (or globally hyperbolic) if, there exists $\lambda \in (0,1)$ such that the tangent vector bundle over M splits into two Df-invariant subbundles $TM = E^u \oplus E^s$, with $\|Df^n|_{E^s}\| \le \lambda^n$ and $\|Df^{-n}|_{E^u}\| \le \lambda^n$. A completely analog definition for symplectomorphisms can be given. We observe that there are plenty Anosov diffeomorphisms which are not volume-preserving and there are plenty Anosov volume-preserving diffeomorphisms which are not symplectic. Anosov was the first one to study these kind if systems when considering the geodesic flow on closed Riemannian manifolds displaying negative curvature ([3]).

Example 1.1 [Arnold's cat map].—The map on the two-torus M, $f\colon M \to M$ defined by

$$f(x,y) = (2x+y, x+y) (\mathrm{mod}\ 1)$$

is an area-preserving diffeomorphism thus, since the manifold is two dimensional also symplectomorphism, on the torus which is Anosov.

It is well-known that Anosov diffeomorphisms display the shadowing property (see e.g. [24]). However, the shadowing property itself do not assure hyperbolicity. Notwithstanding, the stability of the shadowing property allows us to conclude hyperbolicity (cf. Theorem A and Theorem B).

The concept of *structural stability* was introduced in the mid 1930s by Andronov and Pontrjagin ([2]), it led to the construction of uniformly hyperbolic theory, and characterizing, along a tour de force program culminated in the works by Mañé ([16, 17, 18]), structural stability as being essentially equivalent to uniform hyperbolicity. In brief terms it means that under small perturbations the dynamics are topologically equivalent: a dynamical system is C^r-structurally stable if it is topologically conjugated to any other system in a C^r neighbourhood.

Being an Anosov map is very rigid and imposes stringent topological constraints on the manifold. Actually, in the late sixties, Franks proved that the only surfaces that support hyperbolic diffeomorphisms are the tori (see [12]).

Given $f \in \mathrm{Diff}^1_\mu(M)$ (respectively $f \in \mathrm{Symp}^1_\omega(\mathbf{M})$) we say that f is in $\mathscr{F}^1_\mu(M)$ (respectively $\mathscr{F}^1_\omega(\mathbf{M})$) if there exists a neighborhood $\mathscr{V}$ of f in $f \in \mathrm{Diff}^1_\mu(M)$ (respectively f in $f \in \mathrm{Symp}^1_\omega(\mathbf{M})$) such that any $g \in \mathscr{V}$, has all the periodic orbits of hyperbolic type.

Our results ([7]) can be seen as a generalization of the result in [25] for symplectomorphisms and volume-preserving diffeomorphisms. Let us state our first result.

Theorem A.—If $f \in \mathrm{Symp}^1_\omega(\mathbf{M})$ is C^1-stably shadowable, then f is Anosov.

Furthermore, we obtain the analogous version for volume-preserving maps.

Theorem B.—If $f \in \mathrm{Diff}^1_\mu(M)$ is C^1-stably shadowable, then f is Anosov.

As we already said Anosov diffeomorphisms impose severe topological restrictions to the manifold where they are supported. Thus, we present a simple but startling consequence of previous theorems that shows how topological conditions on the phase space imposes numerical restrictions to a given dynamical system.

Corollary 1.2.—If the manifold do not support an Anosov diffeomorphisms, then there are no C^1-stably shadowable symplectomorphisms neither C^1-stably shadowable volume-preserving diffeomorphisms.

We end this introduction by recalling a result in the vein of ours; C^1-robust topologically stable symplectomorphisms are Anosov (see [10]). Another result which relates C^1-robust properties with hyperbolicity is the Horita and Tahzibi theorem (see [13]) which states that C^1-robust transitive symplectomorphisms are partially hyperbolic. We also mention the results in [8, 9] where it is obtained that the stable weak shadowing property implies weak hyperbolicity. Informally speaking weakly shadowing allows that the pseudo-orbits may be approximated by true orbits if one forgets the time parametrization and consider only the distance between the orbit and the pseudo-orbit as two sets in the ambient space.

Moreover, weak hyperbolicity allows the existence of subbundles with neutral behavior.

2. Proof of Theorem A

Theorem A is a direct consequence of the following two propositions. The following result, due to Newhouse, can be found in [22].

Proposition 2.1 ([22]).—If $f \in \mathscr{F}^1_\omega(\mathbf{M})$, then f is Anosov.

Proposition 2.2 is a symplectic version of [19, Proposition 1]. Actually, Moriyasu, while working in the dissipative context, considered the shadowing property in the non-wandering set, which, in the symplectic setting, and due to Poincaré recurrence, is the whole manifold **M**. Let us explain with detail this last step: we say that a point x is non-wandering if any open neighborhood U of x is such that $f^n(U) \cap U \neq \emptyset$ for some $n \in \mathbb{N}$. A point x is said to be recurrent if for any open neighborhood U of x we have $f^n(x) \in U$ for some $n \in \mathbb{N}$. Clearly, every recurrent point is non-wandering. It follows from Poincaré recurrence theorem (see e.g. [15]) that, in our conservative context, we have that μ-a.e. point x is recurrent. Since μ is the Lebesgue measure and the set of non-wandering points is closed, we have that the non-wandering points are the whole manifold **M**.

Proposition 2.2.—If f is a C^1-stably shadowable symplectomorphism, then $f \in \mathscr{F}^1_\omega(\mathbf{M})$.

Proof.—The proof is by reductio ad absurdum; let us assume that there exists a C^1-stably shadowable symplectomorphism f having a non-hyperbolic closed orbit p of period π.

In order to go on with the argument we need to C^1-approximate the symplectomorphism f by a new one, f_1, which, in the local coordinates given by Darboux's theorem, is linear in a neighborhood of the periodic orbit p. To perform this task, in the sympletic setting, and taking into account [5, Lemma 3.9], it is required higher smoothness of the symplectomorphism.

Thus, if f is of class C^∞, take $g = f$, otherwise we use Zehnder's smoothing theorem ([26]) in order to obtain a C^∞ C^1-stably shadowable symplectomorphism h, arbitrarily C^1-close to f, and such that h has a periodic orbit q, close to p, with period π. We observe that q may not be the analytic continuation of p and this is precisely the case when 1 is an eigenvalue of the tangent map $Df^\pi(p)$.

If q is not hyperbolic take $g = h$. If q is hyperbolic for $Dh^\pi(q)$, then, since h is C^1-arbitrarily close to f, the distance between the spectrum of $Dh^\pi(q)$ and the unitary circle can be taken arbitrarily close to zero. This means that we are in the presence of a quite feeble hyperbolicity, thus in a position to apply [12, Lemma 5.1]to obtain a new C^1-stably shadowable symplectomorphism $g \in \text{Symp}^\infty_\omega(M)$, C^1-close to h and such that q is a non-hyperbolic periodic orbit.

At this point, we use the *weak pasting lemma* ([5, Lemma 3.9]) in order to obtain a C^1-stably shadowable symplectomorphism f_1 such that, in local canonical coordinates, f_1 is linear and equal to Dg in a neighborhood of the periodic non-hyperbolic orbit, q. Moreover, the existence of an eigenvalue, σ, with modulus equal to one is associated to a symplectic invariant two-dimensional subspace contained in the subspace $E^c_q \subseteq T_q\mathbf{M}$ associated to norm-one eigenvalues. Furthermore, up to a perturbation using again [12, Lemma 5.1], σ can be taken rational. This fact assures the existence of periodic orbits arbitrarily close to the f_1-orbit of q. Thus, there exists $m \in \mathbb{N}$ such that $f_1^{m\pi}(q)|_{E^c_q} = (Dg^{m\pi})_q|_{E^c_q} = id$ holds, say in an η-neighborhood of q. Recall that, since f_1 has the shadowing property $f_1^{m\pi}$ also has. Therefore, fixing $\epsilon \in (0, \eta/4)$, there exists $\delta \in (0, \epsilon)$ such that every δ-pseudo $f_1^{m\pi}$-orbit $\{x_n\}_n$ is ϵ-traced by some point in **M**. Take y such that $d(y, q) = 3\eta/4$ and a closed δ-pseudo $f_1^{m\pi}$-orbit $\{x_n\}_n$ such that any ball centered in x_i and with radius ϵ is still contained in the η-neighborhood of q, moreover, take $x_0 = q$ and $x_s = y$.

By the shadowing property there exists $z \in \mathbf{M}$ such that $d(f_1^{m\pi i}(z), x_i) < \epsilon$ for any $i \in \mathbb{Z}$. Moreover, we observe that $d(f_1^{m\pi i}(z), q) < \eta$ for every $i \in \mathbb{Z}$. Therefore, $z \in E^c_q$. Finally, we reach a contradiction by noting that

$$d(q,z) \geq d(q,x_s) - d(x_s,z) = \\ = d(q,y) - d(x_s, f_1^{m\pi s}(z)) \geq \frac{3\eta}{4} - \epsilon > \frac{\eta}{2} > \epsilon. \quad \dashv$$

3. Volume-preserving diffeomorphisms

Theorem A also holds on the broader context of volume-preserving diffeomorphisms. Its proof follows the same steps as the one before. The version of Proposition 2.1 for volume-preserving diffeomorphisms was proved in a recent paper by Arbieto and Catalan.

Proposition 3.1 ([4, Theorem 1.1]).—If $f \in \mathscr{F}^1_\mu(M)$, then f is Anosov.

The proof of Theorem B is now reduced to the proof of the following result:

Proposition 3.2.—If f is a C^1-stably shadowable volume-preserving diffeomorphism, then $f \in \mathscr{F}^1_\mu(M)$.

Proof.—Assume that there exists a C^1-stably shadowable $f \in \text{Diff}^1_\mu(M)$ having a non-hyperbolic closed orbit p of period π. Once again we need to C^1-approximate f by a new one, f_1, which, in the local coordinates given by Moser's theorem ([20]), is linear in a neighborhood of the periodic orbit p. Taking into account [5, Theorem 3.6], it is required higher smoothness of the volume-preserving diffeomorphism.

Thus, if f is of class C^∞, take $g = f$, otherwise we use Avila's recent proved smoothing theorem ([6]) in order to obtain a C^∞ C^1-stably shadowable volume-preserving diffeomorphism h, arbitrarily C^1-close to f, and such that h has a periodic orbit q, close to p, with period π.

If q is not hyperbolic take $g = h$. If q is hyperbolic for $Dh^\pi(q)$, then, its weak hyperbolicity allows us to use Franks' lemma proved in [11, Proposition 7.4] for volume-preserving diffeomorphisms and thus obtain a new C^1-stably shadowable volume-preserving diffeomorphism $g \in \text{Diff}^\infty_\mu(M)$, C^1-close to h and such that q is a non-hyperbolic periodic orbit.

Now we use [5, Theorem 3.6] in order to obtain a C^1-stably shadowable volume-preserving diffeomorphism f_1 such that, in local canonical coordinates, f_1 is linear and equal to Dg in a neighborhood of the periodic non-hyperbolic orbit, q. Moreover, the existence of an eigenvalue, σ, with modulus equal to one is associated to an invariant one or two-dimensional subspace contained in the subspace $E^c_q \subseteq T_qM$ associated to norm-one eigenvalues. If its eigendirection is two-dimensional, up to a perturbation using again [11, Proposition 7.4], σ can be taken rational. This fact assures the existence of periodic orbits arbitrarily close to the f_1-orbit of q. Thus, there exists $m \in \mathbb{N}$ such that $f_1^{m\pi}(q)|_{E^c_q} = (Dg^{m\pi})_q|_{E^c_q} = id$ holds, say in a η-neighborhood of q. Finally, we reach a contradiction by arguing exactly as we did in the proof of Theorem A. $\dashv$

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